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ON WEAKLY S-ρ-IDEALS IN NONCOMMUTATIVE RINGS

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Abstract. Let R be a noncommutative ring, and $\phi \neq S \subseteq R$ be an m-system of R. In this paper, we extend the concept of weakly prime ideals in noncommutative rings by introducing weakly S-ρ-ideals. This generalization builds on the notions of right S-prime, weakly S-n-ideals and S-ρ-ideals. We present equivalent definitions and key properties of weakly S-ρ-ideals. We also explore with the method of idealization how the weakly S-ρ-ideal property transfers to related rings and construct such ideals. On the other hand we give a generalization of S-$\mathcal{J}$-ideals which were introduced in [12].

1. Introduction and preliminaries

Let R denote a commutative ring, and let S be a subset of R that is multiplicatively closed. The idea of weakly prime ideals was originally presented in [15] as a generalization of prime ideals. The notion of weakly S-n-ideal introduced in [19] generalizes the concept of S-n-ideals which introduced in [20]. In this context, an ideal P of R is said to be a weakly S-n-ideal if for all $x, y \in R$ with $0 \neq xy \in P$ and $sx \notin \sqrt{0}$, then $sy \in P$, for a fixed $s \in S$.

Let R be a noncommutative ring, S be an m-system of R, and ρ be a special radical. Numerous generalizations have been explored in the noncommutative context, as seen in, for example, [4, 5, 6, 13, 14]. An ideal P of a ring R is considered as a weakly prime ideal if whenever $0 \neq IJ \subseteq P$ implies either $I \subseteq P$ or $J \subseteq P$ for $I, J \lhd R$. The notion of ρ-ideals (weakly ρ-ideals) in noncommutative ring introduced by Nico Groenewald [1] ([3]). Here, a ρ-ideal P (a weakly ρ-ideal) of a ring R is a proper ideal satisfying the condition that if $xRy \subseteq P$ $(0 \neq xRy \subseteq P)$ and $x \notin \rho(R)$, then $y \in P$ for all $x, y \in R$. In [2], the notion of S-ρ-ideal is defined as a generalization of ρ-ideal as follow. An ideal P of R, with $S \cap P = \phi$, is called S-ρ-ideal if and only if whenever $AB \subseteq P$, then either $A\langle s\rangle \subseteq \rho(R)$ or $B\langle s\rangle \subseteq P$ for all $A, B \lhd R$, and for some $s \in S$. Choosing S to be an m-system is essential, as prime ideals are closely associated with m-systems in noncommutative rings, much like their connection with multiplicatively closed subsets in the commutative case. Without this choice, satisfactory results cannot be obtained.

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In this work, we introduce the concept of weakly S-ρ-ideal which is a generalization of S-ρ-ideal in noncommutative rings [2]. We present some equivelant definitions, and many properties. We show in Example 2.6 that our generalization is proper, as the ideal $K = \mathbb{M}_n(\langle 0\rangle)$, of the ring $R = \mathbb{M}_n(\mathbb{Z}_{30})$, is a w-S-ρ-ideal, but not an S-ρ-ideal. In addition, in Example 2.7, we show that the ideal $A = \begin{bmatrix} 0 & \mathbb{R} \\ 0 & 0 \end{bmatrix}$ of the ring $R = \begin{bmatrix} \mathbb{R} & \mathbb{R} \\ 0 & \mathbb{R} \end{bmatrix}$ is a w-S-ρ-ideal which is not a w-ρ-ideal. Proposition 2.12 shows that if P is a weakly S-ρ-ideal but not S-ρ-ideal, then $P^2 = 0$. While in Corollary 2.14 we show that in a local ring (R, M), if M is a weakly S-$\mathcal{P}$-ideal but not S-$\mathcal{P}$-ideal, then, the Jacobson and prime radicals coincide. We also show how the property of being weakly S-ρ-ideal can be transfered to a related ring, and at the end we constract the weakly S-ρ-ideal by using the method of idealization. Also in Proposition 4.5, we give a ring in which each proper ideal is ρ-ideal (weakly ρ-ideal).

In this paper, unless stated otherwise, we focus on rings that are associative, noncommutative, and without identity elements. Furthermore, when we mention an ideal, we are specifically referring to a proper two-sided ideal. The symbols $\mathcal{P}(R)$, $\mathcal{J}(R)$, represent the Baer-McCoy radical and the Jacobson radical of R, respectively.

2. Weakly S-ρ-ideal

In the following, R will denote a ring, S will denote an m-system, and ρ will be a special radical.

Definition 2.1. [Definition 1.1 of [1]] $P \lhd R$ is said to be a ρ-ideal, if for any u, $v \in R$ with $uRv \subseteq P$ and $u \notin \rho(R)$, it follows that $v \in P$.

If P is ρ ideal, then for all $J, K \lhd R$, the inclusion $JK \subseteq P$ implies $J \subseteq \rho(R)$ or $K \subseteq P$. The equivalence of these conditions holds if the ring possesses an identity, as stated in Proposition 1.15 of [1].

Definition 2.2. [Definition 2.2 of [3]] $P \lhd R$ is said to be a weakly ρ-ideal, if for any a, $b \in R$ with $0 \neq aRb \subseteq P$ and $a \notin \rho(R)$, it follows that $b \in P$.

If P is ρ ideal, then for all $A, B \lhd R$, the inclusion $0 \neq AB \subseteq P$ implies either $A \subseteq \rho(R)$ or $B \subseteq P$. The equivalence of these two conditions holds if the ring possesses an identity, as stated in Proposition 2.2 of [3].

The following definition is Definition 2.1 of [2].

Definition 2.3. An ideal P of R, which does not intersect with S, is referred to as an S-ρ-ideal if and only if for any $J, K \lhd R$ such that $JK \subseteq P$, there exists some (fixed) $s \in S$ such that either $J\langle s\rangle \subseteq \rho(R)$ or $K\langle s\rangle \subseteq P$.

In the following, the definition of weakly S-ρ-ideal is given.

Definition 2.4. An ideal K of R, which has no elements in common with S, is termed a weakly S-ρ-ideal if and only if for any non-zero product $AB \subseteq K$, there exists a fixed element $s \in S$ such that either $A\langle s\rangle \subseteq \rho(R)$ or $B\langle s\rangle \subseteq K$, where $A, B \lhd R$.

To avoid lengthiness, we will use the symbol w-S-ρ-ideal to represent weakly S-ρ-ideals. In the following we give an equivalent definitions of w-S-ρ-ideals.

Clearly from the definitions above, one can observe that every S-ρ-ideal is a w-S-ρ-ideal, and every ρ-ideal (w-ρ-ideal) disjoint from S, is also a w-S-ρ-ideal. In addition, if $S \subseteq U(R)$, then, the concepts of w-ρ-ideal and w-S-ρ-ideal coincide. Actually, for an ideal P disjoint from S the following diagram shows the relations clearly:

$$\begin{tikzcd}
\rho\text{-ideal} \arrow[r] \arrow[d] & S\text{-}\rho\text{-ideal} \arrow[d] \\
w\text{-}\rho\text{-ideal} \arrow[r] & w\text{-}S\text{-}\rho\text{-ideal}
\end{tikzcd}$$

Proposition 2.5. *Let $P \lhd R$ where R has an identity. If $P \cap S = \phi$, then, the following are equivalent.*

(1) *P is a w-S-ρ-ideal.*

(2) *For all $x, y \in R$, if $0 \neq \langle x\rangle\langle y\rangle \subseteq P$, then for some $s \in S$, either $\langle x\rangle\langle s\rangle \subseteq \rho(R)$ or $\langle y\rangle\langle s\rangle \subseteq P$.*

(3) *For all $x, y \in R$, if $0 \neq xRy \subseteq P$, there exists an element $s \in S$ such that either $x\langle s\rangle \subseteq \rho(R)$ or $y\langle s\rangle \subseteq P$.*

Proof. (1) $\Rightarrow$ (2) This follows directly from Definition 2.4.

(2) $\Rightarrow$ (3) Suppose $0 \neq xRy \subseteq P$ for some $x, y \in R$. Then, $0 \neq \langle x\rangle\langle y\rangle \subseteq P$, and by (2), there exists $s \in S$, such that either $x\langle s\rangle \subseteq \langle x\rangle\langle s\rangle \subseteq \rho(R)$, or $y\langle s\rangle \subseteq \langle y\rangle\langle s\rangle \subseteq P$.

(3) $\Rightarrow$ (1) Assume $A\langle s\rangle \nsubseteq \rho(R)$ and $B\langle s\rangle \nsubseteq P$ for all $s \in S$. In this case, we can choose $a \in A - \rho(R)$ and $b \in B - P$. For some $a_1 \in \rho(R) \cap A$ and $b_1 \in P \cap B$, we have $(a + a_1)R(b + b_1) \subseteq P$, and since $(a + a_1)\langle s\rangle \nsubseteq \rho(R)$ and $(b + b_1)\langle s\rangle \nsubseteq P$ for any $s \in S$, it implies $(a + a_1)R(b + b_1) = 0$. By considering all possible cases where a_1 and/or b_1 equal zero, we obtain $ab_1 = a_1 b = a_1 b_1 = 0$. Consequently, $ab = 0$. Thus, $AB = 0$, which leads to a contradiction. $\square$

Now we show an example of a w-S-ρ-ideal that is not an S-ρ-ideal.

Example 2.6. Let take a ring $R = \mathbb{M}_n(\mathbb{Z}_{30})$ and an ideal $K = \mathbb{M}_n(\langle 0\rangle)$ of R. S is an m-system such that $S = \{s, s^2, s^4, s^8, ...\}$ where $S = 2(e_{11} + e_{22} + ... + e_{nn})$. Then since R is finite, it is Artinian, hence the Jacobson and the prime radical of R are equal, and by Theorem 10.21 of [10] we know $\mathcal{P}(\mathbb{M}_n(R)) = \mathbb{M}_n(\mathcal{P}(R))$ hence $\mathcal{P}(R) = \mathcal{P}(\mathbb{M}_n(\mathbb{Z}_{30})) = \mathbb{M}_n(\mathcal{P}((\mathbb{Z}_{30})) = \mathbb{M}_n(\langle 0\rangle)$. Thus we obtain $\mathcal{J}(R) = \mathcal{P}(R) = \mathbb{M}_n(\langle 0\rangle)$. Consider the ideals $A = \mathbb{M}_n(\langle 10\rangle), B = \mathbb{M}_n(\langle 3\rangle)$, $AB \subseteq K$ but neither $A\langle s\rangle \nsubseteq \rho(R)$ nor $B\langle s\rangle \nsubseteq K$. Thus K is not an S-ρ-ideal. However, clearly seen that K is a w-S-ρ-ideal of R.

The following example, shows a w-S-ρ-ideal which is not a w-ρ-ideal.

Example 2.7. $R = \begin{bmatrix} \mathbb{R} & \mathbb{R} \\ 0 & \mathbb{R} \end{bmatrix}$, has only the following three proper nonzero two-sided ideals of R are:

$$A = \begin{bmatrix} 0 & \mathbb{R} \\ 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} \mathbb{R} & \mathbb{R} \\ 0 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 0 & \mathbb{R} \\ 0 & \mathbb{R} \end{bmatrix}.$$

Example 2.2 of [3], shows that A is not a w-ρ-ideal, hence is not ρ-ideal. Let $S = \{s\}$, where $s = \begin{bmatrix} 0 & 0 \\ 0 & r \end{bmatrix}$, for some $0 \neq r \in \mathbb{R}$. Since r is a unit element, then S is an m-system of R. In addition, $S \cap A = \phi$. The special radical of R, is:

$$\rho(R) = \begin{bmatrix} \rho(\mathbb{R}) & \mathbb{R} \\ 0 & \rho(\mathbb{R}) \end{bmatrix} = \begin{bmatrix} 0 & \mathbb{R} \\ 0 & 0 \end{bmatrix}.$$

Now assume $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} u_1 & u_2 \\ 0 & u_3 \end{bmatrix} \begin{bmatrix} \mathbb{R} & \mathbb{R} \\ 0 & \mathbb{R} \end{bmatrix} \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix} \subseteq A$. Thus, $[u_1 = 0$ or $a_1 = 0]$ and $[u_3 = 0$ or $a_3 = 0]$, and hence, we have the following four cases: $u_1 = u_3 = 0$, $u_1 = a_3 = 0$, $u_3 = a_1 = 0$, or $a_1 = a_3 = 0$. In all cases we have:

$$\text{either } \begin{bmatrix} u_1 & u_2 \\ 0 & u_3 \end{bmatrix} \langle s \rangle \subseteq \rho(R) \text{ or } \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix} \langle s \rangle \subseteq A.$$

Thus, A is a w-S-ρ-ideal of R.

In referring to [11], for an ideal J of any ring R, $\rho(R) \subseteq \rho^*(J)$ and $J \subseteq \rho^*(J)$, where $\rho^*(J)$ is the radical of the ideal J. Moreover, $\rho(R/J) = \rho^*(J)/J$. In fact, $J \subseteq \rho(R)$ if and only if $\rho(R) = \rho^*(J)$; i.e. if and only if $\rho(R/J) = \rho(R)/J$.

Proposition 2.8. *Let $P \lhd R$ where R has an identity, such that for some $s \in S$ $(\rho(R) : \langle s \rangle) = \rho(R)$. If $P \cap S = \phi$, then, the following are equivalent.*

(1) *P is a w-S-ρ-ideal of R associated with $s \in S$.*

(2) *(a) $P \subseteq (\rho(R) : \langle s \rangle)$.*

(b) Whatever $x, u \in R$, the inclusion $0 \neq xRu \subseteq P$, implies $x \in (\rho^*(P) : \langle s \rangle)$ or $u \in (P : \langle s \rangle)$.

Proof. (1) $\Rightarrow$ (2) (a) Assume $P \nsubseteq (\rho(R) : \langle s \rangle)$, then, for all $x \in P \backslash (\rho(R) : \langle s \rangle)$, $0 \neq xR1 \subseteq P$, hence, by (1) and Proposition 2.5, we obtain $1 \in (P : \langle s \rangle) = R$ for some $s \in S$, a contradiction, because $P \cap S = \phi$.

(b) Now assume for some x, u of R, $0 \neq xRu \subseteq P$. Then, by Proposition 2.5, there exists $s \in S$, such that $x \in (\rho(R) : \langle s \rangle) \subseteq (\rho^*(P) : \langle s \rangle)$ or $u \in (P : \langle s \rangle)$.

(2) $\Rightarrow$ (1) Suppose $0 \neq A_1A_2 \subseteq P$ for $A_1, A_2 \lhd R$. Assume that $A_1\langle s \rangle \nsubseteq \rho(R)$ and $A_2\langle s \rangle \nsubseteq P$ for all $s \in S$. Then, there exist $a_1 \in A_1 \backslash (\rho(R) : \langle s \rangle)$ and $a_2 \in A_2 \backslash (P : \langle s \rangle)$. Hence, $0 \neq a_1Ra_2 \subseteq A_1A_2 \subseteq P$, and by (b) we obtain $a_1 \in (\rho^*(P) : \langle s \rangle)$ or $a_2 \in (P : \langle s \rangle)$. Now since $P \subseteq (\rho(R) : \langle s \rangle) = \rho(R)$, then, $\rho^*(P) = \rho(R)$. Consequently, $a_1 \in (\rho(R) : \langle s \rangle)$ or $a_2 \in (P : \langle s \rangle)$, which leads to contradiction in all cases. Consequently, P is a w-S-ρ-ideal of R. $\square$

In refer to [5], $P \lhd R$ with $P \cap S = \phi$ is called a weakly right S-prime ideal if whenever $J, K \lhd R$ with $0 \neq JK \subseteq P$, implies either $J\langle s \rangle \subseteq P$ or $K\langle s \rangle \subseteq P$. Equivalently, (when R is unitary) for $j, k \in R$ with $0 \neq jRk \subseteq P$, we have $j\langle s \rangle \subseteq P$ or $k\langle s \rangle \subseteq P$.

Corollary 2.9. *Let $P \lhd R$, which has no elements in common with S. If the ideal P is w-S-prime contained in $\rho(R)$, then, P is a w-S-ρ-ideal.*

Proof. Suppose $0 \neq a_1Ra_2 \subseteq P$. If $a_1\langle s\rangle \nsubseteq \rho(R)$ for $a_1, a_2 \in R$, then, $a_1\langle s\rangle \nsubseteq P$. Because P is w-S-prime, we obtain $a_2\langle s\rangle \subseteq P$. Hence, by Proposition 2.5, P is w-S-ρ. □

Theorem 2.10. *Let P be an ideal of a unitary ring R. If P has no elements in common with S, then the following statements are equivalent:*
(1) *P is a w-S-ρ-ideal associated with $s \in S$.*
(2) *For all $r \in R - (P : \langle s\rangle)$ either $(P : \langle r\rangle) \subseteq (\rho(R) : \langle s\rangle)$ or $(P : \langle r\rangle) = (0 : \langle r\rangle)$.*

Proof. (1) ⇒ (2): Let $y \in (P : \langle r\rangle)$, then $y\langle r\rangle \subseteq P$. If $y\langle r\rangle = 0$, then $y \in (0 : \langle r\rangle)$.
If $0 \neq y\langle r\rangle \subseteq P$, then $0 \neq yRr \subseteq P$, and hence by (1) and Proposition 2.5, either $y\langle s\rangle \subseteq \rho(R)$ or $r\langle s\rangle \subseteq P$. Consequently, either $r \in (P : \langle s\rangle)$, contradiction, or $y \in (\rho(R) : \langle s\rangle)$. Thus, $(P : \langle r\rangle) \subseteq (\rho(R) : \langle s\rangle) \cup (0 : \langle r\rangle)$. Hence, either $(P : \langle r\rangle) \subseteq (\rho(R) : \langle s\rangle)$ or $(P : \langle r\rangle) = (0 : \langle r\rangle)$, (recall that $(0 : \langle r\rangle) \subseteq (P : \langle r\rangle)$).
(2) ⇒ (1): Suppose that $0 \neq aRb \subseteq P$ for some $a, b \in R$. If $b\langle s\rangle \nsubseteq P$, then $b \notin (P : \langle s\rangle)$, hence, by (2), either $(P : \langle b\rangle) \subseteq (\rho(R) : \langle s\rangle)$ or $(P : \langle b\rangle) = (0 : \langle b\rangle)$. Since $a \in (P : \langle b\rangle)$, then, either $a \in (0 : \langle b\rangle)$, which implies $aRb = 0$, contradiction, or $a \in (\rho(R) : \langle s\rangle)$, hence, $a\langle s\rangle \subseteq \rho(R)$, consequently, P is a w-S-ρ-ideal. □

In the following, we give other equivalent condition for an ideal to be classified as a w-S-ρ-ideal, and the proof is similar to the proof of Theorem 2.10. Recall that for any $I, J \lhd R$, $(I : J)^* = \{r \in R : Jr \subseteq I\}$ is an ideal of R.

Theorem 2.11. *Let P be an ideal of a unitary ring R. If P has no elements in common with S, then the following are equivalent:*
(1) *P is a w-S-ρ-ideal associated with $s \in S$.*
(2) *For all $r \in R - (\rho(R) : \langle s\rangle)$ either $(P : \langle r\rangle)^* \subseteq (P : \langle s\rangle)$ or $(P : \langle r\rangle)^* = (0 : \langle r\rangle)^*$.*

Proof. (1) ⇒ (2): Let $y \in (P : \langle r\rangle)^*$, then $\langle r\rangle y \subseteq P$. If $\langle r\rangle y = 0$, then $y \in (0 : \langle r\rangle)^*$.
If $0 \neq \langle r\rangle y \subseteq P$, then $0 \neq rRy \subseteq P$, and hence by (1) and Proposition 2.5, either $r\langle s\rangle \subseteq \rho(R)$ that is a contradiction, or $y\langle s\rangle \subseteq P$ consequently, $y \in (P : \langle s\rangle)$. Thus, $(P : \langle r\rangle)^* \subseteq (P : \langle s\rangle) \cup (0 : \langle r\rangle)^*$. Hence, either $(P : \langle r\rangle)^* \subseteq (P : \langle s\rangle)$ or $(P : \langle r\rangle)^* = (0 : \langle r\rangle)^*$.
(2) ⇒ (1): Suppose that $0 \neq aRb \subseteq P$ for some $a, b \in R$. If $a\langle s\rangle \nsubseteq \rho(R)$ for all $s \in S$, then $a \notin (\rho(R) : \langle s\rangle)$, hence, by (2), either $(P : \langle a\rangle)^* \subseteq (P : \langle s\rangle)$ or $(P : \langle a\rangle)^* = (0 : \langle a\rangle)^*$. Since $b \in (P : \langle a\rangle)^*$, then, either $b \in (0 : \langle a\rangle)^*$, which implies $aRb = 0$, contradiction, or $b \in (P : \langle s\rangle)$, hence, $b\langle s\rangle \subseteq P$, consequently, P is a w-S-ρ-ideal. □

Proposition 2.12. *If P is a w-S-ρ-ideal which is not S-ρ, then $P^2 = 0$.*

Proof. Since P is a w-S-ρ-ideal but not an S-ρ-ideal, then, there exist $A_1, A_2 \lhd R$ such that for all $s \in S$, $A_1\langle s\rangle \nsubseteq \rho(R)$ and $A_2\langle s\rangle \nsubseteq P$, but $0 = A_1A_2 \subseteq P$. Now let assume that $P^2 \neq 0$, then, $0 \neq P^2 \subseteq (A_1 + P)(A_2 + P) \subseteq P$. This implies that $(A_1 + P)\langle s_1 \rangle \subseteq \rho(R)$ or $(A_2 + P)\langle s_1 \rangle \subseteq P$ for some $s_1 \in S$. Consequently, we have $A_1\langle s_1 \rangle \subseteq \rho(R)$ or $A_2\langle s_1 \rangle \subseteq P$, which contradicts our assumption. Hence, P^2 must be zero. $\square$

In the following example, we show that the converse of Proposition 2.12 is not true.

Example 2.13. Let ρ be the Baer-McCoy radical of R, and consider $R = T_2(\mathbb{Z}) = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix}$. Let $P = \begin{bmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{bmatrix}$. Then, clearly $P^2 = 0$. Consider S as an m-system of $\mathbb{Z}$. The set

$$S_{T_2(\mathbb{Z})} = \left\{ \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}; s \in S \right\} \text{ also forms an } m\text{-system of } T_2(\mathbb{Z}).$$

Notice that $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} T_2(\mathbb{Z}) \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix} \subseteq P$, and $P \cap S_{T_2(\mathbb{Z})} = \phi$.

However, $\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} T_2(\mathbb{Z}) \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} T_2(\mathbb{Z}) \nsubseteq \mathcal{P}(R) = P$,

and $\begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix} T_2(\mathbb{Z}) \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} T_2(\mathbb{Z}) \nsubseteq P$, for any $s \in S$. Thus, P is not a w-S-ρ-ideal.

It is clear, by Proposition 2.12, that if $P \lhd R$, with $P^2 \neq 0$, then, P is an S-ρ-ideal if and only if P is a w-S-ρ-ideal. Moreover, in case ρ is the Baer-McCoy radical $\mathcal{P}$, if P is a w-S-ρ-ideal but not S-ρ-ideal, then, again by Proposition 2.12, we obtain $P^2 = 0 \subseteq \mathcal{P}(R)$, and hence, $P \subseteq \mathcal{P}(R)$, due to the fact that the Baer-McCoy radical is a semiprime ideal. In the following, we give an interesting property related to a local ring in which its maximal ideal is a w-S-$\mathcal{P}$-ideal but not S-$\mathcal{P}$-ideal.

Corollary 2.14. *Let (R, M) be a local ring, in which M is a w-S-$\mathcal{P}$-ideal but not S-$\mathcal{P}$-ideal. Then, $\mathcal{P}(R) = \mathcal{J}(R)$*

Proof. By Proposition 2.12, $M^2 = 0 \subseteq \mathcal{P}(R)$, hence, $\mathcal{J}(R) = M \subseteq \mathcal{P}(R) \subseteq \mathcal{J}(R)$. $\square$

Corollary 2.15. *Let M be an R-module. If $PM = M$ for a w-S-ρ-ideal P of R which is not S-ρ, then, M is zero.*

Proof. By Proposition 2.12 we get $P^2 = 0$. Hence $M = PM = P^2M = 0$. $\square$

Corollary 2.16. *[Theorem 2.1 of [3]] If P is a w-ρ-ideal which is not a ρ-ideal, then $P^2 = 0$.*

Proof. The proof is derived from Proposition 2.12 by setting $S = 1$. $\square$

Corollary 2.17. *If $\{J_i : i \in \Delta\}$ is a nonempty set of w-S-ρ-ideals of a ring R. Then $\bigcap J_i$ is w-S-ρ.*

Proof. Let $0 \neq a_1Ra_2 \subseteq \bigcap_{i\in\Delta} J_i$ with $a_1\langle s\rangle \nsubseteq \rho(R)$ for $a_1, a_2 \in R$. Then $0 \neq a_1Ra_2 \subseteq J_i$ for each $i \in \Delta$. Because J_i is a w-S-ρ-ideal, we find $a_2\langle s\rangle \subseteq J_i$ for each $i \in \Delta$. Thus $y\langle s\rangle \subseteq \bigcap_{i\in\Delta} J_i$ that is complete the proof. □

Proposition 2.18. *Let R be a unitary ring and $I \lhd R$, such that $I \cap S = \phi$.*

(1) If I is a w-S-ρ-ideal, then $I\langle s\rangle \subseteq \rho(R)$ for some $s \in S$.
(2) $\rho(R)$ is a w-S-ρ-ideal of R if and only if $\rho(R)$ is a w-S-prime ideal of R

Proof. (1) Let $a \in I$. Since $I \cap S = \phi$, $1\langle s\rangle \nsubseteq I$ for all $s \in S$. Hence, $\{0\} \neq aR1 \subseteq I$ implies that there exists an $s \in S$ such that $a\langle s\rangle \subseteq \rho(R)$. Hence $I\langle s\rangle \subseteq \rho(R)$.
(2) Clear. □

In referring to [8], the pseudo radical of an ideal J of a ring R ($\sqrt{J}$), equals to the sum of all $U \lhd R$ such that $U^k \subseteq J$ for some $k \in \mathbb{Z}^+$. Notice that $\sqrt{J} \subseteq \mathcal{P}^*(J)$. While Definition 3.1 (1) of [9], notes that an ideal P of a ring R is said to be S-finite, if there is a finitely generated ideal I such that $P\langle s\rangle \subseteq I \subseteq P$ for some $s \in S$. In the following, we show some connections between w-S-ρ and S-finite ideals. Recall that if $\mathcal{S}_\rho = \{A\colon\ \rho(A) = 0\}$, and $\mathcal{K} = \{A\colon\ A \text{ is a prime ring}\}$, then, $\rho(A) = \cap\{B \lhd A : A/B \in \mathcal{K} \cap \mathcal{S}_\rho\}$, see [7].

Proposition 2.19. *Let $A_1, A_2 \lhd R \in \mathcal{K}$ with $A_1 \nsubseteq (\rho(R) : \langle s\rangle)$ for each $s \in S$. If $\sqrt{A_1A_2}$ is a finitely generated w-S-ρ-ideal, then, $\sqrt{A_2}$ is S-finite.*

Proof. Suppose that $\sqrt{A_1A_2}$ is a w-S-ρ-ideal. Then, by Lemma 1.2 of [8], $\sqrt{A_1}\sqrt{A_2} \subseteq \sqrt{A_1} \cap \sqrt{A_2} = \sqrt{A_1 \cap A_2} = \sqrt{A_1A_2}$. If $\sqrt{A_1}\sqrt{A_2} = 0$, then, either $A_1 \subseteq \sqrt{A_1} = 0 \subseteq (\rho(R) : \langle s\rangle)$, contradiction, or $A_2 \subseteq \sqrt{A_2} = 0$, hence A_2 is S-finite. If $\sqrt{A_1}\sqrt{A_2} \neq 0$, then, by assumption, either $\sqrt{A_1}\langle s\rangle \subseteq \rho(R)$, which implies $A_1 \subseteq (\rho(R) : \langle s\rangle)$, a contradiction, or $\sqrt{A_2}\langle s\rangle \subseteq \sqrt{A_1A_2} = \sqrt{A_1} \cap \sqrt{A_2} \subseteq \sqrt{A_2}$, hence $\sqrt{A_2}$ is S-finite. □

Proposition 2.20. *Let P_1, P_2 be ideals of a unitary ring $R \in \mathcal{K}$ such that $A_1 \nsubseteq \rho(R)$. If $\sqrt{A_1A_2}$ is a finitely generated w-ρ-ideal, then, $\sqrt{A_2}$ is finitely generated and $\sqrt{A_1A_2} = \sqrt{A_2}$.*

Proof. The proof is derived from Proposition 2.19 by setting $S = \{1\}$. □

If S is an m-system of a ring R, then for some ring epimorphism $\psi\colon R \to \psi(R)$, $\psi(S)$ is an m-system of $\psi(R)$ by Lemma 2.1 of [9]. In addition, the set $\bar{S} = \{\bar{s} = s + I : s \in S\}$, is an m-system of R/I for any ideal I of R. In the following we show how the property of being S-ρ-ideal is maintained when transferring to related rings.

Theorem 2.21. *Let $\psi\colon R_1 \to R_2$ be a rings epimorphism, and I be a w-S-ρ-ideal of R_1 such that* $\ker(\psi) \subseteq I$. *Then $\psi(I)$ is a w-$\psi(S)$-ρ-ideal of R_2.*

Proof. Let $0 \neq xR_2y \subseteq \psi(I)$ and $x\langle\psi(s)\rangle \nsubseteq \rho(R_2)$ for $x, y \in R_2$. Because ψ is an epimorphism, we can find $c, d \in R_1$ with $\psi(c) = x$ and $\psi(d) = y$. Hence, $0 \neq xR_2y = \psi(c)\psi(R_1)\psi(d) = \psi(cR_1d) \subseteq \psi(I)$. Since $ker(\psi) \subseteq I$ we get $0 \neq cR_1d \subseteq \psi^{-1}(\psi(I)) = I$. Since I is S-ρ of R_1, either $c\langle s\rangle \subseteq \rho(R_1)$ or

$d\langle s\rangle \subseteq I$ for some $s \in S$. If $c\langle s\rangle \subseteq \rho(R_1)$, then $x\langle\psi(s)\rangle = \psi(c)\langle\psi(s)\rangle \subseteq \psi(\rho(R_1))$. Since ρ is a complete, idempotent Hoehnke radical, then $\psi(\rho(R_1)) \subseteq \rho(R_2)$. Hence, $\psi(c)\langle\psi(s)\rangle \subseteq \rho(R_2)$ which is a contradiction. Thus, $c\langle s\rangle \nsubseteq \rho(R_1)$, hence, $d\langle s\rangle \subseteq I$. Since $y = \psi(d)$, then $y\langle\psi(s)\rangle \subseteq \psi(I)$. Thus, $\psi(I)$ is a w-$\psi(S)$-ρ-ideal of R_2. □

Corollary 2.22. *Let $\psi : R_1 \to R_2$ be a ring epimorphism, and $B \lhd R_2$ such that $\psi^{-1}(B)$ is a w-S-ρ-ideal of R_1. Then, B is a w-$\psi(S)$-ρ-ideal of R_2.*

Proof. As the inverse image of any ideal of R_2 is an ideal of R_1 contains $\ker(\psi)$, the proof follows from Theorem 2.21. □

Theorem 2.23. *Let ψ: $R_1 \to R_2$ be a rings monomorphism, and $P \lhd R$ such that $\ker(\psi) \subseteq P \cap \rho(R_1)$ and $P \cap S = \phi$. If $\psi(P)$ is a w-$\psi(S)$-ρ ideal of R_2, then P is a w-S-ρ-ideal of R_1.*

Proof. Suppose that $0 \neq A_1B_1 \subseteq P$ for $A_1, B_1 \lhd R$. Then, $0 \neq \psi(A_1)\psi(B_1) = \psi(A_1B_1) \subseteq \psi(P)$. Since $\psi(P)$ is a w-$f(S)$-ρ ideal of R_2, then either $\psi(B_1)\langle\psi(s)\rangle \subseteq \psi(P)$, or $\psi(A_1)\langle\psi(s)\rangle \subseteq \rho(R_2)$, for some $s \in S$.

If $\psi(B_1)\langle\psi(s)\rangle \subseteq \psi(P)$, then $B_1\langle s\rangle \subseteq \psi^{-1}(\psi(B_1\langle s\rangle) \subseteq \psi^{-1}(\psi(P)) = P$, since $\ker\psi \subseteq P$. If $\psi(A_1)\langle\psi(s)\rangle \subseteq \rho(R_2)$, then, let $J \lhd R_1$ such that $R_1/J \in \mathcal{S}_\rho \cap \mathcal{K}$, then $\rho(R_1) \subseteq J$, thus $\ker(\psi) \subseteq J$. Hence, $\psi(R_1)/\psi(J) \cong (R_1/\ker(\psi))/(J/\ker\psi) \cong R_1/J$, so $R_2/\psi(J) = \psi(R_1)/\psi(J) \in \mathcal{S}_\rho \cap \mathcal{K}$, and hence $\psi(A_1)\langle\psi(s)\rangle \subseteq \rho(R_2) \subseteq \psi(J)$, thus $A_1\langle s\rangle \subseteq \psi^{-1}(\psi(A_1\langle s\rangle)) \subseteq \psi^{-1}(\psi(J)) = J$, for all J in $\cap\{I \lhd R : R/I \in \mathcal{S}_\rho \cap \mathcal{K}\}$, consequently, $A_1\langle s\rangle \subseteq \rho(R_1)$. Thus, P is a w-S-ρ-ideal of R_1. □

Corollary 2.24. *Let $\psi : R_1 \to R_2$ be a ring monomorphism, and B be a w-$\psi(S)$-ρ ideal of R_2. If $\ker(\psi) \subseteq \rho(R)$, then $\psi^{-1}(B)$ is a w-S-ρ-ideal of R_1.*

Proof. Let $P = \psi^{-1}(B)$. Then, P is a w-S-ρ-ideal of R_1. by Theorem 2.23, since $\ker(\psi) \subseteq P \cap \rho(R)$ and $\psi(P) = \psi(\psi^{-1}(B)) = B$ is a w-$\psi(S)$-ρ ideal of R_2. □

Theorem 2.25. *Let $A, B \lhd R$ such that $A \subseteq B$.*

(1) If B is a w-S-ρ-ideal of R, $\overline{S}$-ρ-ideal of R/A.
(2) If B/A is a w-$\overline{S}$-ρ-ideal of R/A.
Then (1) implies (2). The converse is hold if A is a w-S-ρ ideal of R.

Proof. (1) $\Rightarrow$ (2) Suppose B is a w-S-ρ-ideal of R, then $B \cap S = \phi$, and hence, $A \cap S = \phi$, which implies that $(B/A) \cap \overline{S} = \phi$. Take the canonical epimorphism ψ: $R \to R/A$, since, $\ker(\psi) = A \subseteq B$, then, by Theorem 2.21 we obtain that $\psi(B) = B/A$ is a w-$\overline{S}$-ρ-ideal of R/A.

(2) $\Rightarrow$ (1) We establish that (2) $\Rightarrow$ (1) given the supplementary assumption that A is w-S-ρ. Let $a, b \in R$ such that $\{0\} \neq aRb \subseteq B$. If $(a + A)R/A(b + A) = \{\overline{0}\}$, then $\{0\} \neq aRb \subseteq A$. There exists $s \in S$ such that $a\langle s\rangle \subseteq \rho(R)$ or $b\langle s\rangle \subseteq A \subseteq B$. Now, assume that $\{\overline{0}\} \neq (a + A)R/A(b + A) \subseteq B/A$. Now, since B/A is a w-$\overline{S}$-ρ-ideal of R/A there exists $\overline{s} \in \overline{S}$ such that $a + A\langle\overline{s}\rangle \subseteq \rho(R/A)$ or $b + A\langle\overline{s}\rangle \subseteq B/A$. By an argument similar to that in the proof of Theorem 2.19 we get $a\langle s\rangle \subseteq \rho(R)$ or $b\langle s\rangle \subseteq B$, as desired. □

Let S_1, S_2 be two m-systems of R with $S_1 \subseteq S_2$, then, clearly, every w-S_1-ρ-ideal of R disjoint from S_2, is a w-S_2-ρ-ideal. However, the converse is not true in general. In the following we show that the converse holds under special condition.

Proposition 2.26. *Let S_1, S_2 be two m-systems of R with $S_1 \subseteq S_2$, and for every $a \in S_2$ there exists $b \in S_2$ such that $arb \in S_1$ for some $r \in R$. If P is a w-S_2-ρ-ideal of R, then P is a w-S_1-ρ-ideal.*

Proof. Assume $0 \neq xRy \subseteq P$. Then $x\langle s_1\rangle \subseteq \rho(R)$ or $y\langle s_1\rangle \subseteq P$ for some $s_1 \in S_2$. By assumption, there exists some $s_2 \in S_2$ such that $s = s_1 r s_2 \in S_1$ for some $r \in R$. Hence, $x\langle s\rangle \subseteq x\langle s_1\rangle\langle s_2\rangle \subseteq \rho(R)$ or $y\langle s\rangle \subseteq y\langle s_1\rangle\langle s_2\rangle \subseteq P$. □

Proposition 2.27. *Let P be a w-S-ρ-ideal of R. If $I \lhd R$ meets S, then we have the following:*

(1) *$I \cap P$ is a w-S-ρ-ideal of R.*
(2) *PI is a w-S-ρ-ideal of R.*

Proof. (1) Let $0 \neq AB \subseteq (P \cap I)$ for some $A, B \lhd R$. Then, $0 \neq AB \subseteq P$, hence, either $A\langle s\rangle \subseteq \rho(R)$ or $B\langle s\rangle \subseteq P$ for some $s \in S$. If $B\langle s\rangle \subseteq P$, then for some $x \in (I \cap S)$, $BRsRxR \subseteq (P \cap I)$. Since s, x in the m-system S, there exists $r \in R$ such that $s_1 = srx$, and hence $B\langle s_1\rangle = BRs_1R \subseteq BRsRxR \subseteq (P \cap I)$. If $A\langle s\rangle \subseteq \rho(R)$, then similarly, we obtain $A\langle s_1\rangle \subseteq \rho(R)$. Hence,$(P \cap I)$ is a w-S-ρ-ideal associated with $s_1 \in S$, since $(P \cap I) \cap S = \phi$.

(2) Similar to (1). □

Proposition 2.28. *Let K, J be w-S-ρ-ideals of a ring R. If $(K + J) \cap S = \phi$, then $K + J$ is a w-S-ρ-ideal of R.*

Proof. Let $\bar{S}_1 = \{s + (K \cap J) : s \in S\}$ and $\bar{S}_2 = \{s + J : s \in S\}$, then both of $\bar{S}_1$ and $\bar{S}_2$ are m-systems of $R/(K \cap J)$ and R/J, respectively. Hence, by Theorem 2.25, $K/(K \cap J)$ is a w-$\bar{S}_1$-ρ-ideal of $R/(K \cap J)$. Since $\bar{S}_1 \subseteq \bar{S}_2$, then $K/(K \cap J)$ is also a w-$\bar{S}_2$-ρ-ideal of $R/(K \cap J)$ by the remark made above Proposition 2.26. Thus, $(K+J)/J$ is a w-S-ρ-ideal of R/J becuase $(K+J)/J \cong K/(K\cap J)$. Thus, by Theorem 2.25, we conclude that $K + J$ is a w-S-ρ-ideal of R. □

Definition 2.29. Let I be a w-S-ρ-ideal ideal of R. We say (a, b) is an s -twin-zero of I if $aRb = 0$, $a\langle s\rangle \nsubseteq \rho(R)$, and $b\langle s\rangle \nsubseteq I$ for every $s \in S$.

Note that if I is a w-S-ρ-ideal ideal of R that is not a S-ρ-ideal ideal of R, then I has an s- twin-zero (a, b) for some $a, b \in R$.

Corollary 2.30. *Let P be a w-S-ρ-ideal of R and assume $AB \subseteq P$ for some $A, B \lhd R$. If P has an s-twin zero (a, b) for some $a \in A$ and $b \in B$, then $AB = 0$.*

Proof. Assume $0 \neq AB \subseteq P$, since P is a w-S-ρ-ideal, then, either $A\langle s\rangle \subseteq \rho(R)$, or $B\langle s\rangle \subseteq P$ for some $s \in S$. Let (a, b) be the s-twin zero for some $a, b \in R$ with $a \in A$ and $b \in B$. Now by the definition of s-twin zero $a\langle s\rangle \nsubseteq \rho(R)$ and $b\langle s\rangle \nsubseteq P$, contradiction. Hence $AB = 0$. □

It is well known that if the rings R_1, R_2 are with identities, then any ideal of $R_1 \times R_2$ has the form $A \times B$, where A and B are ideals of R_1 and R_2, respectively. In addition, $\mathcal{P}(R_1 \times R_2) = \mathcal{P}(R_1) \times \mathcal{P}(R_2)$, see Lemma 3 of [18]. Moreover, if S_1 is an m-system of R_1 and S_2 is an m-system of R_2, then, $S_1 \times S_2$ is an m-system of the ring $R_1 \times R_2$. It is easy to check that for an ideal A of R_1, $(A \times R_2) \cap (S_1 \times S_2) = \phi$, if and only if $A \cap S_1 = \phi$.

Theorem 2.31. *Let $R = R_1 \times R_2$, where R_1 and R_2 are rings with identities, and let $S = S_1 \times S_2$, where S_1 and S_2 are m-systems of R_1 and R_2, respectively. Let P_1 be a nonzero ideal of R_1 such that , $P_1 \times R_2$ is a w-S-$\mathcal{P}$-ideal, then, P_1 is a w-S_1-$\mathcal{P}$-ideal and $\mathcal{P}(R_2) \cap S_2 \neq \phi$.*

Proof. Suppose that $P = P_1 \times R_2$ is a w-S-$\mathcal{P}$-ideal of R. Let $0 \neq a \in P_1$, then, $(0,0) \neq (a,1) \in P$, hence, $(0,0) \neq (a,1)R(1,1) \subseteq P$. By assumption, there exists $(s_1,s_2) \in S$ such that either $(a,1)\langle(s_1,s_2)\rangle \subseteq \mathcal{P}(R_1 \times R_2) = \mathcal{P}(R_1) \times \mathcal{P}(R_2)$ hence $\langle s_2\rangle \subseteq \mathcal{P}(R_2)$, or $(1,1)\langle(s_1,s_2)\rangle \subseteq P$ which implies $\langle s_1\rangle \subseteq P_1$, contradiction. Thus, $\mathcal{P}(R_2) \cap S_2 \neq \phi$. Now let $0 \neq IJ \subseteq P_1$ for some $I, J \lhd R_1$, then, $0 \neq (I \times 0)(J \times 0) \subseteq P$, and hence, by assumption, either $(I \times 0)\langle(s_1,s_2)\rangle \subseteq \mathcal{P}(R_1) \times \mathcal{P}(R_2)$ or $(J \times 0)\langle(s_1,s_2)\rangle \subseteq P$, consequently, either $I\langle s_1\rangle \subseteq \mathcal{P}(R_1)$ or $J\langle s_1\rangle \subseteq P_1$. Since $P \cap S = \phi$, we get $P_1 \cap S_1 = \phi$. Thus, P_1 is an S_1-$\mathcal{P}$-ideal of R_1. □

Theorem 2.32. *Let $R = R_1 \times R_2$, where R_1 and R_2 are rings with identities, and let $S = S_1 \times S_2$, where S_1 and S_2 are m-systems of R_1 and R_2, respectively. Let P_2 be a nonzero ideal of R_2 such that $R_1 \times P_2$ is a w-S-$\mathcal{P}$-ideal, then, P_2 is a w-S_2-$\mathcal{P}$-ideal and $\mathcal{P}(R_1) \cap S_1 \neq \phi$.*

Proof. Similar to the proof of Theorem 2.31. □

Corollary 2.33. *Let $R = R_1 \times R_2$, where R_1 and R_2 are rings, and let $S = S_1 \times S_2$ be an m-system of R, where S_1 and S_2 are m-systems of R_1 and R_2, respectively. If $P = P_1 \times P_2$ is a w-S-$\mathcal{P}$-nonzero ideal of R for some ideals P_1, P_2 of R_1 and R_2, respectively, then, ${P_1}^2 = 0$ and ${P_2}^2 = 0$. Particularly, if $P_1 \times R_2$ ($R_1 \times P_2$) is a w-S-$\mathcal{P}$-ideal, then, ${R_2}^2 = 0$ (${R_1}^2 = 0$).*

Proof. By Proposition 2.11 of [3], there is no nonzero $\mathcal{P}$-ideal of R, hence, by Proposition 2.12, $P^2 = 0$ and thus, ${P_1}^2 = 0$ and ${P_2}^2 = 0$. □

3. Idealization

We now show how to construct w-S-ρ-ideal ideals using the method of idealization. In the following, M is an $R-R$-bimodule. The idealization of M is the ring $R \boxplus M$ with $(R \boxplus M, +) = (R, +) \oplus (M, +)$ and the multiplication is given by $(r,m)(s,n) = (rs, rn + ms)$. $R \boxplus M$ itself is, in a canonical way, an $R - R$ -bimodule and $M \simeq 0 \boxplus M$ is a nilpotent ideal of $R \boxplus M$ of index 2. We also have $R \simeq R \boxplus 0$ and the latter is a subring of $R \boxplus M$. Note also that $R \boxplus M$ is a subring of the Morita ring $\begin{bmatrix} R & M \\ 0 & R \end{bmatrix}$ via the mapping $(r,m) \mapsto \begin{bmatrix} r & m \\ 0 & r \end{bmatrix}$. We will require some knowledge about the ideal structure of $R \boxplus M$. If $I \lhd R$ and N is an $R-R$- bi-submodule of M, then $I \boxplus N$ is an ideal of $R \boxplus M$ if and only if $IM + MI \subseteq N$.

It is easy to check that if S is an m-system of the ring R, then $S \boxplus M$ is an m-system of the ring $R \boxplus M$.

Theorem 3.1. *Let M be an $R-R-$bimodule and I a proper ideal of R. Then:*

(1) *If $I \boxplus N$ is a w-S-ρ-ideal of $R \boxplus M$ and $RMR \subseteq N$; then I is a w-S-ρ-ideal of R :*

(2) *$I \boxplus M$ is a w-$S \boxplus M$-ρ-ideal of $R \boxplus M$ if and only if I is a w-S-ρ-ideal of R and for any $s-$twin zero (a, b) of I we have $aM = Mb = 0$.*

Proof. (1) If $I = R$; then $I \boxplus N = R \boxplus M$; a contradiction. Let $r_1; r_2 \in R$ with $\{0\} \neq r_1 R r_2 \subseteq I$ and $r_1 \langle s \rangle \nsubseteq \rho(R)$ for $s \in S$: Then, since $RMR \subseteq N$ we have $\{(0;0)\} \neq (r_1;0)R \boxplus M(r_2;0) \subseteq I \boxplus N$ and $(r_1, 0) \langle (s;0) \rangle \nsubseteq \rho(R \boxplus M) = \rho(R) \boxplus M$: Since $I \boxplus N$ is a w-S-ρ-ideal of $R \boxplus M$; we have $(r_2, 0) \langle (s;0) \rangle \subseteq I \boxplus N$. Hence $r_2 \langle s \rangle \subseteq I$ as needed.

(2) Suppose $I \boxplus M$ is a w-$S \boxplus M$-ρ-ideal of $R \boxplus M$. Let $0 \neq aRb \subseteq I$ where $a, b \in R$. Now $(0, 0) \neq (a, 0)R \boxplus M(b, 0) \subseteq I \boxplus M$ and $I \boxplus M$ a w-$S \boxplus M$-ρ-ideal gives $(a, 0) \langle (s, m) \rangle \subseteq \rho(R \boxplus M) = \rho(R) \boxplus M$ or $(b, 0) \langle (s, m) \rangle \subseteq I \boxplus M$ for some $(s, m) \in S \boxplus M$. Now $(a, 0) \langle (s, m) \rangle = (a, 0)R \boxplus M\,(s, m)\,R \boxplus M \subseteq a \langle s \rangle \boxplus M \subseteq \rho(R) \boxplus M$ or $(b, 0) \langle (s, m) \rangle \subseteq b \langle s \rangle \boxplus M \subseteq I \boxplus M$. Hence $a \langle s \rangle \subseteq \rho(R)$ or $b \langle s \rangle \subseteq I$ for some $s \in S$. So I is a w-S-ρ -ideal. Assume (a, b) is an s-twin zero of I. We claim that $aM = Mb = 0$. Suppose $aM \neq 0$, so there is $m \in M$ such that $am \neq 0$. Thus we get $(0, 0) \neq (a, 0)(1, 0)(b, m) \in (a, 0)R \boxplus M(b, m) \subseteq aRb \boxplus M = 0 \boxplus M \subseteq I \boxplus M$. But $(a, 0) \langle (s, m) \rangle = (a \langle s \rangle, MsR + RmR + RsM) \nsubseteq \rho(R) \boxplus M$ and $(b, m) \langle (s, m) \rangle \nsubseteq I \boxplus M$ for each $(s, m) \in S \boxplus M$, contradicting the fact that $I \boxplus M$ is a w-$S \boxplus M$-ρ-ideal.

Conversely, suppose $(0, 0) \neq (a, m)R \boxplus M(b, n) \subseteq I \boxplus M$ for $a, b \in R$ and $n, m \in M$. We get $aRb \subseteq I$. The possible two cases are follow:

Case 1: $0 \neq aRb \subseteq I$. Now I a w-S-ρ-ideal of R gives $a \langle s \rangle \subseteq \rho(R)$ or $b \langle s \rangle \subseteq I$ for some $s \in S$. Thus $(a, m) \langle (s, 0) \rangle = (a, m)\,(RsR, MsR + RsM) \subseteq (a \langle s \rangle, M) \subseteq \rho(R) \boxplus M$ or $(b, n) \langle (s, 0) \rangle \subseteq I \boxplus M$ for $(s, 0) \in S \boxplus M$, as required.

Case 2: $0 = aRb \subseteq I$. Suppose $a \langle s \rangle \nsubseteq \rho(R)$ and $b \langle s \rangle \nsubseteq I$ for each $s \in S$. Thus (a, b) is an S-twin zero of I and from assumption $aM = Mb = 0$. Now $(a, m)R \boxplus M(b, n) = (aRb, aM + aMb + Mb) = (0, 0)$ a contradiction . □

Taking $S = \{1\}$ in the above Theorem 3.1, we get the following. From [3, Proposition 2.9].

Corollary 3.2. *(1)If $I \boxplus N$ is a w-ρ-ideal of $R \boxplus M$ and $RMR \subseteq N$; then I is a w-ρ-ideal of R:*

(2)Let M be an $R-R-$bimodule and $I \lhd R$. Then $I \boxplus M$ is a w-ρ-ideal of $R \boxplus M$ if and only if I is a w-ρ-ideal of R and for any twin zero (a, b) of I we get $aM = Mb = 0$.

Corollary 3.3. *Let (R, E) be a local ring and M be an $R-R$ bi-module such that $EM = 0$. Then $I \boxplus M$ is a w-$S \boxplus M$-ρ-ideal of $R \boxplus M$ if and only if I is a w-S-ρ- ideal of R.*

Proof. Let $a, b \in R$ with $aRb = 0$, but $a \langle s \rangle \nsubseteq \rho(R)$ and $b \langle s \rangle \nsubseteq I$ for every $s \in S$. Assume that $a \notin E$ or $b \notin E$. Without losing generality, we may suppose that $a \notin E$. Then there is $c \in R$ such that $ac = 1$ and thus $acb = b \in aRb = 0$, a contradiction. So, $a \in E = Ann(M)$ and $b \in E = Ann(M)$. By Theorem 3.1 the result is obtained. □

4. Weakly S-$\mathcal{J}$-ideals

4.1. **In commutative rings.** In the following the ring R will be a commutative ring, and S denote a subset of R that is closed under multiplication.

The concept of S-$\mathcal{J}$-ideals in commutative rings was introduced in [12]. According to Definition 2.1 of [12], an ideal K of R that is disjoint from S is called an S-$\mathcal{J}$-ideal if there exists a fixed element $s \in S$ such that for all $x, y \in R$, whenever $xy \in I$, either $sx \in \mathcal{J}(R)$ or $sy \in K$.

In the following we generalize the above notion by introducing the notion of weakly S-$\mathcal{J}$-ideals.

Definition 4.1. An ideal K of R is said to be a weakly S-$\mathcal{J}$-ideal, if whenever $0 \neq xy \in K$, for $x, y \in R$, either $sx \in \mathcal{J}(R)$ or $sy \in K$ for some (fixed) $s \in S$.

It is clear that the zero ideal is always a weakly S-$\mathcal{J}$-ideal, and every S-$\mathcal{J}$-ideal is also a weakly S-$\mathcal{J}$-ideal, but the converse is not true as we show in the following examples.

Example 4.2. Let $R = \mathbb{Z}_{30}$. It is obvious that $\mathcal{J}(R) = \langle\overline{0}\rangle$. $S = \{\overline{1}, \overline{2}, \overline{4}, \overline{8}, \overline{16}\}$ is a multiplicatively closed subset of R. For $s = \overline{2} \in S$, $0 \neq \overline{3}.\overline{10} \in \langle\overline{0}\rangle$ but neither $s\overline{3} \in \mathcal{J}(R)$ nor $s\overline{10} \in \langle\overline{0}\rangle$ Hence $\langle\overline{0}\rangle$ is not an S-$\mathcal{J}$-ideal. But it is clear that zero ideal is a weakly S-$\mathcal{J}$-ideal.

4.2. **In noncommutative rings.** In [1], Groenewald introduced the notion of ρ ideals as a generalization of n-ideals and $\mathcal{J}$-ideals. In the following the ring R will be a noncommutative ring, and S will refer to an m-system of R.

Definition 4.3. An ideal K of R is said to be a right weakly S-$\mathcal{J}$-ideal, if whenever $0 \neq IJ \in K$, for ideals I, J of R, either $I\langle s\rangle \subseteq \mathcal{J}(R)$ or $J\langle s\rangle \subseteq K$ for some (fixed) $s \in S$.

Example 4.4. Let $R = \mathbb{Z} \boxplus (\mathbb{Z}_2 \times \mathbb{Z}_2)$. $S = S_1 \boxplus (S_2 \times S_2)$ is a multiplicatively closed subset of R where $S_1 = \{1\}, S_2 = \{0, 1\}$. The ideal $K = 0 \boxplus \langle(\overline{1}, \overline{0})\rangle$ of R is not an S-$\mathcal{J}$-ideal of R since for an element $s = (1, (\overline{1}, \overline{1}))$, while $(2, (\overline{0}, \overline{0}))(0, (\overline{1}, \overline{1})) = (0, (\overline{0}, \overline{0}))$, neither $s(2, (\overline{0}, \overline{0})) \in \mathcal{J}(R)$ nor $s(0, (\overline{1}, \overline{1})) \in K$. But from the Example 2.2 of [16], we know that K is a weakly $\mathcal{J}$-ideal hence K is a weakly S-$\mathcal{J}$-ideal.

In the following we give a proposition to show that in a noncommutative Artinian local ring $\mathcal{A}$ each proper ideal is a ρ-ideal. From [17, Proposition 2.3] in a commutative local ring $\mathcal{A}$ each proper ideal is a $\mathcal{J}$-ideal and in [3] in a local noncommutative ring $\mathcal{A}$ each proper ideal is a $\mathcal{J}$-ideal. Since every $\mathcal{J}$-ideal is an S-$\mathcal{J}$-ideal, and weakly S-$\mathcal{J}$-ideal then we could say that in a local ring (commutative or noncommutative) $\mathcal{J}$-ideal, S-$\mathcal{J}$-ideal, weakly $\mathcal{J}$-ideal and S-$\mathcal{J}$-ideal coincide.

Proposition 4.5. *Let R be a noncommutative ring. If R is an Artinian local ring, then each proper ideal of R is a ρ-ideal.*

Proof. Let P be an ideal and $x, y \in R$ with $xRy \in P$ and $x \notin \rho(R) = \mathcal{J}(R) = M$ where M is a unique maximal ideal of R. Then $x \in U(R)$ and so $x^{-1}xRy = 1Ry \subseteq P$. Thus $y \in P$ and P is an ρ-ideal.

□

Proposition 4.6. [17, Proposition 2.3]*Let R be a commutative ring. The following is equivalent.*
1. *R is a local ring.*
2. *Each proper ideal of R is a $\mathcal{J}$-ideal..*

Proposition 4.7. [3, Theorem 5.6]*Let R be a noncommutative ring. The following is equivalent.*
1. *R is a local ring.*
2. *Each proper ideal of R is a $\mathcal{J}$-ideal.*

Theorem 4.8. *Let R be a commutative ring and P be a weakly S-$\mathcal{J}$-ideal that is not an S-$\mathcal{J}$-ideal. Then $P^2 = 0$.*

Proof. Assume $P^2 \neq 0$. Let $xy \in P$ and $sx \notin \mathcal{J}(R)$ for $x, y \in R, s \in S$. If $xy \neq 0$, then $sy \in P$ since P is a weakly S-$\mathcal{J}$-ideal. Assume $xy = 0$. If $xP \neq 0$, then $0 \neq xt$ for some $t \in P$ and thus $0 \neq x(y+t) \in P$. Since P is a weakly S-$\mathcal{J}$-ideal, then $(y+t)s \in P$. Hence $ys \in P$. If $yP \neq 0$, then $yk \neq 0$ for some $k \in P \subseteq \mathcal{J}(R)$. Since $0 \neq ky = (k+x)y \in P$. Clearly $(k+x)s \notin \mathcal{J}(R)$, then $ys \in P$. Thus assume that $xP = yP = 0$. Since $P^2 \neq 0$, then there is $a, b \in P$, such that $ab \neq 0$. Therefore $0 \neq ab = (a+x)(b+y) \in P$ and $s(a+x) \notin \mathcal{J}(R)$ imply $(b+y)s \in P$. Hence $ys \in P$. Thus P is an S-$\mathcal{J}$-ideal. □

Corollary 4.9. [16, Theorem 2.5]*Let R be a commutative ring. If P is a weakly $\mathcal{J}$-ideal that is not a $\mathcal{J}$-ideal, then $P^2 = 0$.*

Proof. By taking $S = \{1\}$ in Theorem 4.8, the result is obtained. □

Corollary 4.10. *If P is a weakly S-$\mathcal{J}$-ideal but not an S-$\mathcal{J}$-ideal, then $P \subseteq \mathcal{P}(R)$.*

Proof. Clear by Theorem 4.8. □

Corollary 4.11. *Let $\mathbb{M}$ be a R-module and P be a weakly S-$\mathcal{J}$-ideal. If P is not an S-$\mathcal{J}$-ideal, then the equation $P\mathbb{M} = \mathbb{M}$ implies $\mathbb{M} = 0$.*

Proof. Let $P\mathbb{M} = \mathbb{M}$. Then $\mathbb{M} = P\mathbb{M} = P^2\mathbb{M} = 0$ by Theorem 4.8. □

Conflict of interest. The authors declare no conflicts of interest

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