



On the networks of large embeddings

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Abstract. We define a special network that exhibits the large embeddings in any class of similar algebras. With the aid of this network, we introduce a notion of distance that conceivably counts the minimum number of dissimilarities, in a sense, between two given algebras in the class in hand; with the possibility that this distance may take the value ∞ . We display a number of inspirational examples from different areas of algebra, e.g., group theory and monounary algebras, to show that this research direction can be quite remarkable.

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1. Introduction

We assume familiarity with the most basic conceptions of universal algebra; the notions of algebra, subalgebra, homomorphism, isomorphism, embedding, direct product, etc. A modest perceptiveness to some special areas of algebra is also required, e.g., group theory, fields, lattices, Boolean algebras, etc. This paper is written with the appetite of reducing the technical difficulties and the passion of creating a text accessible to mathematicians with general background. There is a considerable number of illustrative figures that can demonstrate some concepts and ideas of proofs.

Throughout, unless otherwise stated, we assume that K is an arbitrary but fixed class of similar algebras. We define a special network that exhibits the large embeddings in the class K . An algebra $\mathfrak{A}$ is largely embeddable into another algebra $\mathfrak{B}$ in K iff there is a member of K that serves as an intermediate algebra in the following sense: It is isomorphic to $\mathfrak{A}$, and it is a large subalgebra of $\mathfrak{B}$. The word ‘large’ here means that this intermediate algebra needs at most

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one extra element to generate the whole of $\mathfrak{B}$. Similar networks can be found in [16, 17].

Investigating this network seems to be thought-provoking and promising to be fruitful. For instance, with the aid of this network, we introduce a notion of distance that conceivably counts the minimum number of dissimilarities, in a sense, between two given structures in K ; with the possibility that this distance may take the value ∞ . Formally, this distance counts the minimum number of steps needed from the symmetric closure of the large embedding relation to reach an algebra from another algebra. This notion of distance provides a framework that can give a qualitative and quantitative analysis of the connections between the algebraic structures in the class K .

Investigating such networks and distances gives an understanding of how and how much non-isomorphic elements of a given class K differ in terms of generating elements within K .

In this paper, we commence a research project that aims to investigate the network and the distance notions for different classes of algebras. We give several examples from different areas of algebra, e.g., group theory, lattice theory, etc. As an example, we give a complete characterization of the network of all subgroups of the group $\mathbb{Q}/\mathbb{Z}$. The last section is completely devoted to the network of monounary algebras. For instance, we give an algorithmic method to determine the distance between any two finite monounary algebras. Similar investigations concerning Boolean algebras and other algebras of logic are burgeoning [16, 15, 27].

We foresee that this open-end research direction would appeal to mathematicians from different disciplines, and that many captivating results can be obtained. We are also discerned with the potential applications that this project may activate. Here is an example: If we suppose that K is the class of all Lindenbaum–Tarski algebras of first-order theories, then the notion of distance we propose here brings a method to determine the concepts distinguishing two theories in hand. This idea could be used to conclude that only one concept distinguishes classical and relativistic kinematics, namely the existence of observers who are at absolute rest, see [17, 19]. This is indeed an interesting result for logicians, philosophers and physicists alike.

It is customary in universal algebra to assume that the underlying set of an algebra is not empty. We follow this tradition here, and thus we have no empty algebras. We usually use the German capital letters $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \dots$ to denote algebras, and we use the corresponding Latin capital letters $A, B, C, \dots$ to denote their underlying sets. However, we may violate this rule when we give examples from areas of algebra where such convention is not used, e.g., group theory. We use the notation $\mathfrak{A} \subseteq \mathfrak{B}$ to convey that $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$. For an algebra $\mathfrak{A}$ and a subset $X \subseteq A$, we let $\langle X \rangle$ denote the subalgebra of $\mathfrak{A}$ generated by X . We consider 0 as a natural number and denote the set of natural numbers by ω .

2. The networks of large embeddings

We start with the central notion of the present paper, the notion of *large subalgebras*. This notion was inspected in some other resources in the literature, cf. [28].

Definition 2.1. Suppose that $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$. We say that $\mathfrak{A}$ is a *large subalgebra* in $\mathfrak{B}$ iff there is an element $b \in B$ such that $\langle A \cup \{b\} \rangle = \mathfrak{B}$.

For an example, Table 1 on page 4 characterizes all the large subgroups of the symmetric group S_4 . This table also shows that a large proper subalgebra is not necessarily maximal among proper subalgebras, e.g., $\{e, (12)\}$ is large but not maximal in S_4 . However, the converse is always true; every subalgebra which is maximal among proper subalgebras is large. The table also shows an example of two isomorphic subgroups, namely $\{e, (12)\}$ and $\{e, (12)(34)\}$, where the former is large in S_4 and the latter is not large in S_4 .

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Now, we construct a special network which is convened from the large inclusions in K . This network is a special case of the cluster networks defined in [17, Section 3]. The *network $\mathcal{N}(K)$ of large embeddings in K* is defined to be the graph whose vertices are the members of K , and which has two types of edges: red dashed edges connecting the isomorphic members of K and blue edges connecting two algebras of K if one of them is a large subalgebra of the other one. (We note that the definition of the network $\mathcal{N}(K)$ is not formalizable in Zermelo–Fraenkel set theory if K is a proper class. However, all our results can be formulated within von Neumann–Bernays–Gödel set theory (NBG), or simply one can use a Grothendieck universe.)

Every vertex in the network $\mathcal{N}(K)$ has two loops; one is red dashed, and the other is blue. When we illustrate networks of large embeddings, we will omit these loops for the lucidity of the illustrations.

Example 2.2. Suppose that K is the class of all subalgebras of the 8-element Boolean algebra $\mathfrak{A}$ having atoms a, b and $c = \neg(a \vee b)$. Then the network of K is illustrated in Figure 1.

Example 2.3. Suppose that K is the class of all subgroups of the alternating group A_4 . Then the network $\mathcal{N}(K)$ is illustrated in Figure 2.

A *path* in the network $\mathcal{N}(K)$ is a finite sequence of edges such that any two consecutive edges share a common vertex. The *blue-length of a path* is defined to be the number of its blue edges. Two algebras $\mathfrak{A}$ and $\mathfrak{B}$ are said to be *connected in $\mathcal{N}(K)$* if there is a path connecting them.

Example 2.4. If K is the class of all finite fields, then two finite fields are connected in $\mathcal{N}(K)$ iff they have the same characteristic.

Definition 2.5. The *generator distance* $d_g: K \times K \rightarrow \omega \cup \{\infty\}$ is defined on the class K as follows: Let $\mathfrak{A}$ and $\mathfrak{B}$ be two algebras in K .

- If $\mathfrak{A}$ and $\mathfrak{B}$ are not connected in $\mathcal{N}(K)$, then $d_g(\mathfrak{A}, \mathfrak{B}) \stackrel{\text{def}}{=} \infty$.

- Otherwise, if $\mathfrak{A}$ and $\mathfrak{B}$ are connected in $\mathcal{N}(\mathbf{K})$, then

$$d_g(\mathfrak{A}, \mathfrak{B}) \stackrel{\text{def}}{=} \min \{ l \mid \text{there is a path of blue-length } l \\ \text{connecting } \mathfrak{A} \text{ and } \mathfrak{B} \text{ in the network } \mathcal{N}(\mathbf{K}) \}.$$

TABLE 1. Large subgroups of the symmetric group S_4

n	Subgroups of S_4 of order n	$\cong$ type	Large in S_4 ?
1	$\{e\}$	Z_1	NO
2	$\{e, (12)\}$ $\{e, (13)\}$ $\{e, (14)\}$ $\{e, (23)\}$ $\{e, (24)\}$ $\{e, (34)\}$ $\{e, (12)(34)\}$ $\{e, (13)(24)\}$ $\{e, (14)(23)\}$	Z_2	YES
3	$\{e, (123), (132)\}$ $\{e, (124), (142)\}$ $\{e, (134), (143)\}$ $\{e, (234), (243)\}$	Z_3	YES
4	$\{e, (12), (34), (12)(34)\}$ $\{e, (13), (24), (13)(24)\}$ $\{e, (14), (23), (14)(23)\}$ $\{e, (1324), (12)(34), (1423)\}$ $\{e, (1234), (13)(24), (1432)\}$ $\{e, (1243), (14)(23), (1342)\}$ $\{e, (12)(34), (13)(24), (14)(23)\}$	$Z_2 \times Z_2$ Z_4	YES YES
6	$\{e, (123), (132), (12), (13), (23)\}$ $\{e, (124), (142), (12), (14), (24)\}$ $\{e, (134), (143), (13), (14), (34)\}$ $\{e, (234), (243), (23), (24), (34)\}$	$Z_2 \times Z_2$ S_3	NO YES
8	$\{e, (12), (34), (12)(34), (13)(24),$ $(14)(23), (1324), (1423)\}$ $\{e, (13), (24), (13)(24), (12)(34),$ $(14)(23), (1234), (1432)\}$ $\{e, (14), (23), (14)(23), (12)(34),$ $(13)(24), (1243), (1342)\}$	D_4	YES
12	$\{e, (12)(34), (13)(24), (14)(23),$ $(123), (132), (124), (142), (134),$ $(143), (234), (243)\}$	A_4	YES
24	S_4	S_4	YES

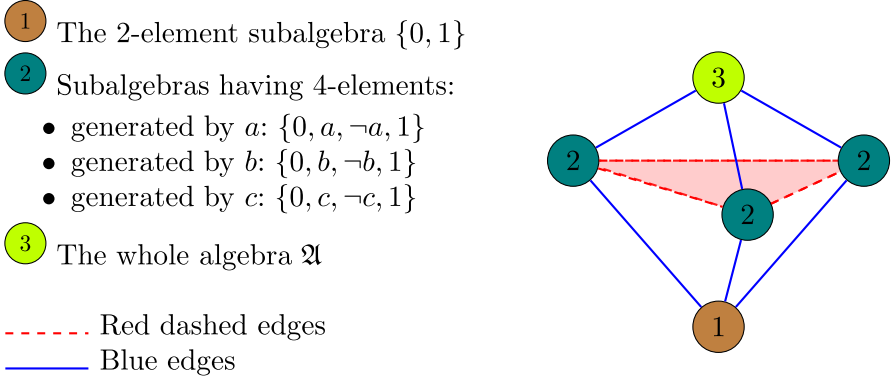


FIGURE 1. A small network of Boolean algebras

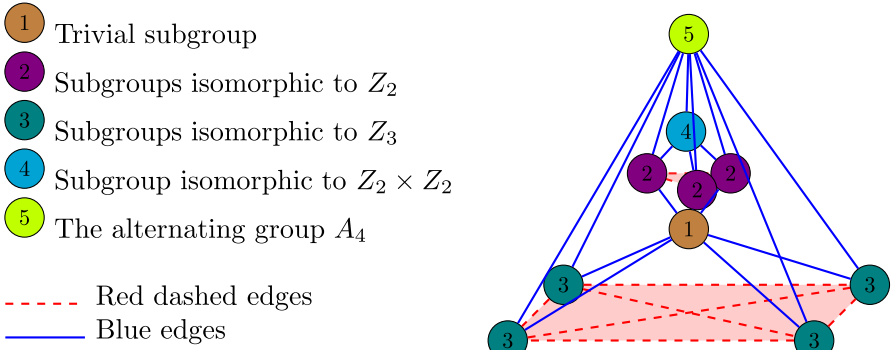


FIGURE 2. A small network of groups

Another way to understand the above definition: Each blue edge represents an actual step, a step of adding or deleting a generator. The red dashed edges represent null steps, steps that are not in our consideration; moving from an algebra to an isomorphic copy represents no real ‘algebraic’ change. The generator distance then counts the minimum number of actual steps needed to reach an algebra starting from another one.

Example 2.6. Suppose that K is the class of all vector spaces over a given field, then the generator distance between two vector spaces V and W in K is the following:

$$d_g(V, W) = \begin{cases} 0 & \text{if } V \cong W, \\ |\dim(V) - \dim(W)| & \text{if } V \text{ and } W \text{ are finite dimensional, and} \\ \infty & \text{otherwise.} \end{cases}$$

Example 2.7. Suppose that K is the class of all groups, and let V_4 and Q_8 denote the Klein four group and the quaternion group, respectively. It is obvious that $d_g(V_4, Q_8) \geq 2$ because none of these groups is isomorphic to or

embeddable into the other one. We claim that the equality holds. To see this, let G be the central product of the dihedral group D_8 and the cyclic group Z_4 . It is not hard to check that each one of V_4 and Q_8 is isomorphic to a large subgroup of G .

Proposition 2.8. *The following are true for every $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in K$:*

- (1) $d_g(\mathfrak{A}, \mathfrak{B}) \geq 0$, and $d_g(\mathfrak{A}, \mathfrak{B}) = 0 \iff \mathfrak{A} \cong \mathfrak{B}$.
- (2) $d_g(\mathfrak{A}, \mathfrak{B}) = d_g(\mathfrak{B}, \mathfrak{A})$.
- (3) $d_g(\mathfrak{A}, \mathfrak{B}) \leq d_g(\mathfrak{A}, \mathfrak{C}) + d_g(\mathfrak{C}, \mathfrak{B})$, where addition with ∞ is defined in the natural way.

Proof. Straightforward. □

The following simple observation is worthy of note. Suppose that L is a subclass of K . The values of the generator distance in L must take bigger or equal values to the corresponding values of the generator distance in K . This is true because every path in $\mathcal{N}(L)$ is a path in $\mathcal{N}(K)$, but the later network may contain extra paths that may reduce the values of the generator distance.

Definition 2.9. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two algebras in K . We say that $\mathfrak{A}$ is *largely embeddable* into $\mathfrak{B}$ in K , and we write $\mathfrak{A} \xrightarrow{(\bullet)} \mathfrak{B}$ or $\mathfrak{B} \xleftarrow{(\bullet)} \mathfrak{A}$, iff there is $\mathfrak{C} \in K$ such that $\mathfrak{A} \cong \mathfrak{C}$ and $\mathfrak{C}$ is a large subalgebra in $\mathfrak{B}$.

There are subgroups that are not large in S_4 , but every non-trivial subgroup of S_4 is largely embeddable into S_4 , see Table 1. In [8, Theorem A], it was shown that, if $n \neq 4$ and $x \in S_n$ is not the identity element, then there is $y \in S_n$ such that x and y generates the whole S_n . Consequently, every non-trivial subgroup of S_n is large if $n \neq 4$. Now, suppose that K is the class of all groups, and let G and H be two finite groups.

- The generator distance between G and H in K is at most 2. This is true because there is always a (large enough) natural number n such that both G and H are embeddable into the symmetric group S_n . Whence, by [8, Theorem A] and the triangle inequality, it follows that

$$d_g(G, H) \leq d_g(G, S_n) + d_g(S_n, H) \leq 1 + 1 = 2.$$

This bound is sharp by Example 2.7.

- For the same reasons, one can see that the distance between G and the trivial group is at most 3. This bound is also sharp and that can be confirmed, for example, by checking that the quaternion group is of distance 3 from the trivial group.

In the illustrative diagrams of this section, a black edge between two algebras means that one of these algebras is largely embeddable into the other one. When we are interested in the direction of the large embedding (which one of the algebras is largely embeddable into the other one) we will use a black arrow instead of the black edge. Again, we discard all the loops.

Assume that the class K is closed under the formation of subalgebras. Thus, the generator distance between two algebras is equal to the minimum length of all paths of black edges that connect these algebras. To find a path

of black edges of length k , for some $k \in \omega$, we have 2^k -many possible options for the arrows (regardless of the different options for the vertices); which makes the problem of determining the generator distance quite difficult. The following properties may reduce this difficulty substantially. Indeed, with these properties, one needs to consider only $k + 1$ -many options for the directions of the arrows to find a path of length k .

Definition 2.10. We say that the class K has the *push-up property* iff for all $\mathfrak{A}, \mathfrak{B} \in K$, we have

$$(\exists \mathfrak{C} \in K) \quad \mathfrak{A} \xleftarrow{(\bullet)} \mathfrak{C} \xrightarrow{(\bullet)} \mathfrak{B} \implies (\exists \mathfrak{D} \in K) \quad \mathfrak{A} \xrightarrow{(\bullet)} \mathfrak{D} \xleftarrow{(\bullet)} \mathfrak{B}. \quad (2.1)$$

We say that the class K has the *push-down property* iff for all $\mathfrak{A}, \mathfrak{B} \in K$, we have

$$(\exists \mathfrak{D} \in K) \quad \mathfrak{A} \xrightarrow{(\bullet)} \mathfrak{D} \xleftarrow{(\bullet)} \mathfrak{B} \implies (\exists \mathfrak{C} \in K) \quad \mathfrak{A} \xleftarrow{(\bullet)} \mathfrak{C} \xrightarrow{(\bullet)} \mathfrak{B}. \quad (2.2)$$

To state the next proposition, we will use the following abbreviation: For any algebras $\mathfrak{A}$ and $\mathfrak{B}$ in the class K , and $m \in \omega$, we write $\mathfrak{A} \xrightarrow{(\bullet)}_{\times m} \mathfrak{B}$ iff there are $\mathfrak{A}_0, \dots, \mathfrak{A}_m$ such that

$$\mathfrak{A} \cong \underbrace{\mathfrak{A}_0 \xrightarrow{(\bullet)} \mathfrak{A}_1 \xrightarrow{(\bullet)} \dots \xrightarrow{(\bullet)} \mathfrak{A}_m}_{m\text{-many arrows}} \cong \mathfrak{B}.$$

Proposition 2.11. Suppose that K is closed under formation of subalgebras. Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are two algebras connected in $\mathcal{N}(K)$. The following are true:

(1) If K has the push-up property, then

$$(\exists \mathfrak{D} \in K)(\exists n, m \in \omega) \quad \mathfrak{A} \xrightarrow{(\bullet)}_{\times m} \mathfrak{D}, \mathfrak{B} \xrightarrow{(\bullet)}_{\times n} \mathfrak{D} \text{ and } d_g(\mathfrak{A}, \mathfrak{B}) = n + m.$$

(2) If K has the push-down property, then

$$(\exists \mathfrak{C} \in K)(\exists n, m \in \omega) \quad \mathfrak{C} \xrightarrow{(\bullet)}_{\times m} \mathfrak{A}, \mathfrak{C} \xrightarrow{(\bullet)}_{\times n} \mathfrak{B} \text{ and } d_g(\mathfrak{A}, \mathfrak{B}) = n + m.$$

Proof. The statements follow from the definitions by straightforward induction on $d_g(\mathfrak{A}, \mathfrak{B})$ which is finite because $\mathfrak{A}$ and $\mathfrak{B}$ are connected in $\mathcal{N}(K)$. See an illustrative proof in Figure 4. $\square$

We recall an important property from the literature. The class K is said to have the *amalgamation property* iff for every $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in K$ and every embeddings $f: \mathfrak{C} \rightarrow \mathfrak{A}$ and $g: \mathfrak{C} \rightarrow \mathfrak{B}$ there is $\mathfrak{D} \in K$ and embeddings $f': \mathfrak{A} \rightarrow \mathfrak{D}$ and $g': \mathfrak{B} \rightarrow \mathfrak{D}$ such that $f' \circ f = g' \circ g$. For more on different versions of the amalgamation property and their applications, we refer the reader to [18].

One should not confuse the push-up property with the amalgamation property. The arrows in Figure 3a represent large embeddings, not arbitrary embeddings, and the commutativity of the diagram in Figure 3a is not required. These are the exact differences between the push-up property and the amalgamation property.

Proposition 2.12. Suppose that K is closed under the formation of subalgebras, then

$$K \text{ has the amalgamation property} \implies K \text{ has the push-up property}.$$

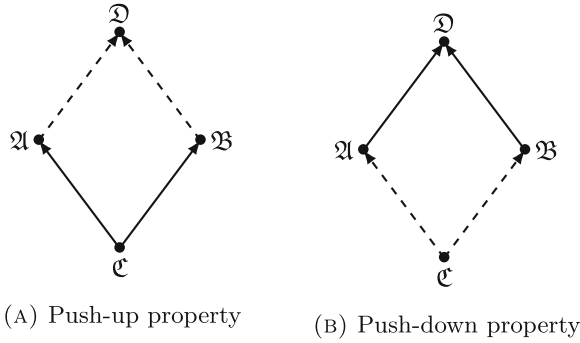


FIGURE 3. Pushing properties

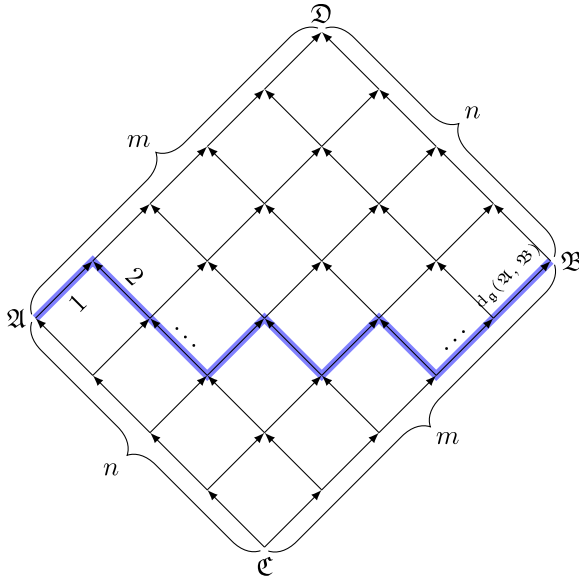


FIGURE 4. Pushing a path upwards and/or downwards

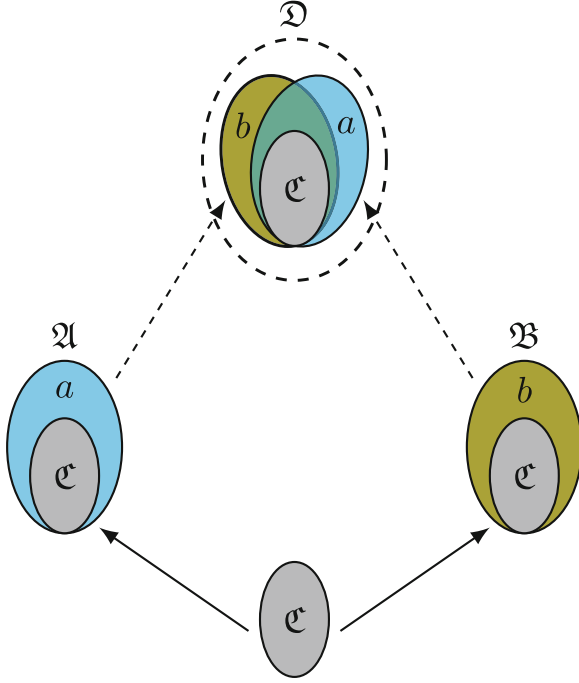
Proof. Assume that $\mathbf{K}$ has the amalgamation property. We need to show that $\mathbf{K}$ has the push-up property. To this end, let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \mathbf{K}$ and suppose that $\mathfrak{C}$ is largely embeddable into both $\mathfrak{A}$ and $\mathfrak{B}$. Thus, there are two embeddings $f: \mathfrak{C} \rightarrow \mathfrak{A}$ and $g: \mathfrak{C} \rightarrow \mathfrak{B}$, $a \in A$ and $b \in B$ such that

$$\mathfrak{A} = \langle f[C] \cup \{a\} \rangle \quad \text{and} \quad \mathfrak{B} = \langle g[C] \cup \{b\} \rangle,$$

here, $f[X]$ denotes the f -image of set X , i.e. $f[X] = \{f(x) \mid x \in X\}$.

Now, by the amalgamation property, there is an algebra $\mathfrak{D} \in \mathbf{K}$ and embeddings $f': \mathfrak{A} \rightarrow \mathfrak{D}$ and $g': \mathfrak{B} \rightarrow \mathfrak{D}$ such that $f'[f[C]] = g'[g[C]] \stackrel{\text{def}}{=} X$. Consider the following subalgebras of $\mathfrak{D}$:

$$\mathfrak{D}_1 = \langle X \cup \{f'(a)\} \rangle, \quad \mathfrak{D}_2 = \langle X \cup \{g'(b)\} \rangle \quad \text{and} \quad \mathfrak{D}' = \langle X \cup \{f'(a), g'(b)\} \rangle.$$

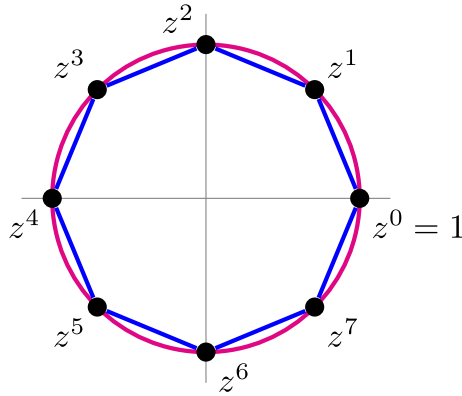
FIGURE 5. Amalgamation property $\implies$ push-up property

Since K is closed under formation of subalgebras, we guarantee that $\mathfrak{D}_1$, $\mathfrak{D}_2$ and $\mathfrak{D}'$ are all members of K . It is not hard to see that $\mathfrak{A} \cong \mathfrak{D}_1$ and $\mathfrak{B} \cong \mathfrak{D}_2$, and both $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are large subalgebras in $\mathfrak{D}'$. Therefore, we conclude that the class K has the push-up property. See Figure 5. $\square$

There are several classes of algebras that have the amalgamation property, and hence the push-up property, such as groups [26, 22], (distributive) lattices [12, 6, 25], Boolean algebras [5], some classes of Boolean algebras with operators [20, 23, 24]. Thus, the difficulty of the problem of determining the values of the generator distance in these classes can be reduced. In what follows, we give an example of how Proposition 2.11 and Proposition 2.12 can be effectively used in this direction. We also note that some other classes do not have the amalgamation property, such as semigroups [4, Section 9.4] and modular lattices [13, 6], and so Proposition 2.12 cannot be useful in these cases.

Consider the group $G \stackrel{\text{def}}{=} \mathbb{Q}/\mathbb{Z}$ and suppose that K is the class of all subgroups of this group. It is known that G is isomorphic to the torsion subgroup of the unit circle. In other words, G is isomorphic to the group that consists of all p^n -th complex roots of the unit, for each prime number p and each $n \in \omega$, and whose operation is the multiplication of complex numbers. See Figure 6. The group G has many interesting properties, we list some of them below.

- (A) G is Abelian and torsion. The later means that each element of G has a finite order.

FIGURE 6. The 2^3 -th complex roots of the unit

- (B) Every finitely generated subgroup of G is cyclic.
- (C) Any two isomorphic subgroups of G must be equal. Thus, K has the amalgamation property, since the group G itself can be always chosen as the amalgamating algebra.
- (D) Let p be a prime number and let $n \in \omega$. The group G contains a cyclic group of order p^n , denoted by $Z(p^n)$. These cyclic subgroups form a chain whose limit gives another important subgroup of G . This subgroup is called the Prüfer p -group and is denoted by $Z(p^\infty)$.

$$Z(p^0) \subseteq Z(p^1) \subseteq Z(p^2) \subseteq Z(p^3) \subseteq \dots \subseteq Z(p^\infty).$$

Let $p_0, p_1, \dots, p_j, \dots$ be an enumeration of the primes in their natural order. Consider an infinite sequence of the form $\bar{k} = (k_0, k_1, \dots, k_j, \dots)$ where each k_i is either a non-negative integer or the symbol ∞ . We call such a sequence a *choice sequence*. Let

$$\langle \bar{k} \rangle = \left\{ g_1 g_2 \cdots g_m \in G \mid 0 \neq m \in \omega \text{ and } \{g_1, g_2, \dots, g_m\} \subseteq \bigcup_{i=0}^{\infty} Z(p_i^{k_i}) \right\}.$$

Obviously, $\langle \bar{k} \rangle$ is a subgroup of G . Moreover, it is not hard to see that the following is true, cf. [1].

- (E) Every subgroup of G is equal to $\langle \bar{k} \rangle$ for some choice sequence $\bar{k} = (k_0, k_1, \dots, k_j, \dots)$.

Let $\bar{k}$ and $\bar{k}'$ be two choice sequences. We need the following notations.

- We write $\bar{k} \equiv \bar{k}'$ iff the following two conditions are satisfied:
 - (1) For each $i \in \omega$, $k_i = \infty \iff k'_i = \infty$, and
 - (2) the set $\{i \in \omega \mid k_i \neq k'_i\}$ is finite.
- We write $\bar{k} \preceq \bar{k}'$ iff (1) $\bar{k} \equiv \bar{k}'$ and (2) $k_i \leq k'_i$, for each $i \in \omega$.

Now, we are ready to give a complete characterization of the network $\mathcal{N}(K)$ of large embeddings and the generator distance in the class K of all subgroups of $G = \mathbb{Q}/\mathbb{Z}$.

Lemma 2.13. *Let $\bar{k}$ and $\bar{k}'$ be two choice sequences.*

- (1) *If $\bar{k} = \bar{k}'$, then $d_g(\langle \bar{k} \rangle, \langle \bar{k}' \rangle) = 0$.*
- (2) *If $\bar{k} \preceq \bar{k}'$ and $\bar{k} \neq \bar{k}'$, then $d_g(\langle \bar{k} \rangle, \langle \bar{k}' \rangle) = 1$.*
- (3) *If $\bar{k} \equiv \bar{k}'$, $\bar{k} \not\preceq \bar{k}'$ and $\bar{k}' \not\preceq \bar{k}$, then $d_g(\langle \bar{k} \rangle, \langle \bar{k}' \rangle) = 2$.*
- (4) *If $\bar{k} \not\equiv \bar{k}'$, then $d_g(\langle \bar{k} \rangle, \langle \bar{k}' \rangle) = \infty$.*

Proof. Let $\bar{k}$ and $\bar{k}'$ be two choice sequences.

- (1) If $\bar{k} = \bar{k}'$, then $\langle \bar{k} \rangle = \langle \bar{k}' \rangle$ and consequently $d_g(\langle \bar{k} \rangle, \langle \bar{k}' \rangle) = 0$.
- (2) Suppose that $\bar{k} \preceq \bar{k}'$ and $\bar{k} \neq \bar{k}'$. Then, $\langle \bar{k} \rangle \subseteq \langle \bar{k}' \rangle$, indeed, for each $i \in \omega$, $Z(p^{k_i}) \subseteq Z(p^{k'_i})$. Remember, the set $\Delta \stackrel{\text{def}}{=} \{i \in \omega \mid k_i \neq k'_i\}$ is finite. For each $j \in \Delta$, let a_j be a generator of the group $Z(p^{k'_j})$. Let

$$a \stackrel{\text{def}}{=} \prod_{j \in \Delta} a_j.$$

Now, it is not hard to see that adding a to $\langle \bar{k} \rangle$ would generate the bigger group $\langle \bar{k}' \rangle$, which means that $\langle \bar{k} \rangle$ is a large subgroup in $\langle \bar{k}' \rangle$ and the desired follows.

- (3) Suppose that $\bar{k} \equiv \bar{k}'$, $\bar{k} \not\preceq \bar{k}'$ and $\bar{k}' \not\preceq \bar{k}$. That means there are $i, j \in \omega$ such that $k_i < k'_i$ and $k_j > k'_j$. Thus, one can find an element of order $p^{k'_j}$ in $\langle \bar{k} \rangle$ but not in $\langle \bar{k}' \rangle$. Similarly, one can find an element of order p^{k_i} in $\langle \bar{k}' \rangle$ but not in $\langle \bar{k} \rangle$. Hence, none of the groups $\langle \bar{k} \rangle$ and $\langle \bar{k}' \rangle$ can be embedded into the other one. That implies $d_g(\langle \bar{k} \rangle, \langle \bar{k}' \rangle) \geq 2$. On the other hand, the set $\{i \in \omega \mid k_i \neq k'_i\}$ is finite. Define a choice sequence $\bar{l} = (l_0, l_1, \dots)$ as follows: for each $i \in \omega$, let $l_i = \max\{k_i, k'_i\}$. Therefore, by the triangle inequality and item (2) above,

$$d_g(\langle \bar{k} \rangle, \langle \bar{k}' \rangle) \leq d_g(\langle \bar{k} \rangle, \langle \bar{l} \rangle) + d_g(\langle \bar{l} \rangle, \langle \bar{k}' \rangle) = 1 + 1 = 2.$$

- (4) Suppose that $\bar{k} \not\equiv \bar{k}'$, then we have two cases.

Case 1: Suppose that there is an $i \in \omega$ such that $k_i = \infty$ but $k'_i \neq \infty$. Assume towards a contradiction that $d_g(\langle \bar{k} \rangle, \langle \bar{k}' \rangle) < \infty$. By property (C) above, we know that the class K has the amalgamation property, and whence, by Proposition 2.12, K has the push-up property too. So, by Proposition 2.11, there are $n, m \in \omega$ and a group $H \subseteq G$ such that $\langle \bar{k} \rangle \xrightarrow{(\bullet)} \times_m H$, $\langle \bar{k}' \rangle \xrightarrow{(\bullet)} \times_n H$ and $d_g(\langle \bar{k} \rangle, \langle \bar{k}' \rangle) = n + m$. We also note that $n \neq 0$ since the group $\langle \bar{k} \rangle$ cannot be embedded into the other group $\langle \bar{k}' \rangle$. Now we consider the embedding $\langle \bar{k}' \rangle \xrightarrow{(\bullet)} \times_n H$.

Note that property (C) implies that the group $\langle \bar{k}' \rangle$ is actually a subgroup of H . Moreover, there are $x_1, \dots, x_n \in G$ such that $\langle \bar{k}' \rangle$ together with $x_1, \dots, x_n$ generates the whole H . We can assume that none of these elements is a member of $\langle \bar{k}' \rangle$; otherwise we would get $d_g(\langle \bar{k} \rangle, \langle \bar{k}' \rangle) < n + m$ which is a contradiction. By the commutativity of G , it follows that H is the internal product of $\langle \bar{k}' \rangle$ and the subgroup $\langle x_1, \dots, x_n \rangle$ generated by $x_1, \dots, x_n$. Now, the property (B) implies that $\langle x_1, \dots, x_n \rangle$ is a cyclic group. Thus, there is an element $x \in G$ such that H is the internal product of $\langle \bar{k}' \rangle$ and the cyclic group $\langle x \rangle$ generated by the element x . Let

$$l = \max(\{k'_i\} \cup \{s \mid p_i^s \text{ divides the order of } x\}).$$

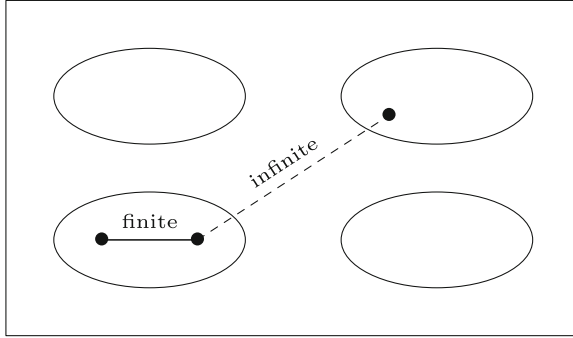


FIGURE 7. Connected components of a network

Since all these groups are Abelian, the order of every element in H is equal to the least common multiple of the orders of an element in $\langle \bar{k}' \rangle$ and an element in $\langle x \rangle$. Thus, H cannot contain an element of order p_i^{l+1} . But $\langle \bar{k} \rangle$ contains an element of order p_i^{l+1} . This contradicts the embedding $\langle \bar{k} \rangle \xrightarrow{(\bullet)}_{\times m} H$. Whence, the desired follows.

Case 2: Suppose that $\{i \in \omega \mid k_i \neq k'_i\}$ is infinite. Without loosing generality, we can assume that $k_i > k'_i$ holds for infinitely many $i \in \omega$. By the same argument (by contradiction) as in Case 1, there has to be an element $x \in G$ such that $\langle \bar{k} \rangle \xrightarrow{(\bullet)}_{\times m} H$ holds for the internal product H of $\langle \bar{k}' \rangle$ and $\langle x \rangle$. From the infinitely many, there has to be one, say j , such that $k_j > k'_j$ and the order of x is not divisible by p_j . Therefore, H cannot contain an element of order $p_j^{k_j}$ contradicting that $\langle \bar{k} \rangle$ can be embedded to H . $\square$

We note that the relation $\equiv$ is an equivalence relation on the set of all choice sequences. Let $\bar{k}$ be a choice sequence and let

$$K(\bar{k}) = \{ \langle \bar{k}' \rangle \subseteq G \mid \bar{k}' \equiv \bar{k} \}.$$

By Lemma 2.13, the values of the generator distance between the members of $K(\bar{k})$ and each other must be finite, while the generator distance between a subgroup in $K(\bar{k})$ and another subgroup outside $K(\bar{k})$ is ∞ . That makes the subclass $K(\bar{k})$ a connected component of $\mathcal{N}(K)$. By a *connected component* of a network of large embeddings, we mean a maximal subclass in which any two algebras are connected in the network.

The *diameter* of a connected component of a network of large embeddings is defined to be the smallest $n \in \omega \cup \{\infty\}$ for which $d_g(\mathfrak{A}, \mathfrak{B}) \leq n$ for all $\mathfrak{A}$ and $\mathfrak{B}$ in the component. So, if we assume again that K is the class of all subgroups of $G = \mathbb{Q}/\mathbb{Z}$, then here are the diameters of the components of the network $\mathcal{N}(K)$: Let $k = (k_0, k_1, \dots, k_j, \dots)$ be a choice sequence.

- If $|\{i \in \omega \mid k_i \neq \infty\}| = 0$, then the diameter of $K(\bar{k})$ is 0.
- If $|\{i \in \omega \mid k_i \neq \infty\}| = 1$, then the diameter of $K(\bar{k})$ is 1.
- If $|\{i \in \omega \mid k_i \neq \infty\}| > 1$, then the diameter of $K(\bar{k})$ is 2.

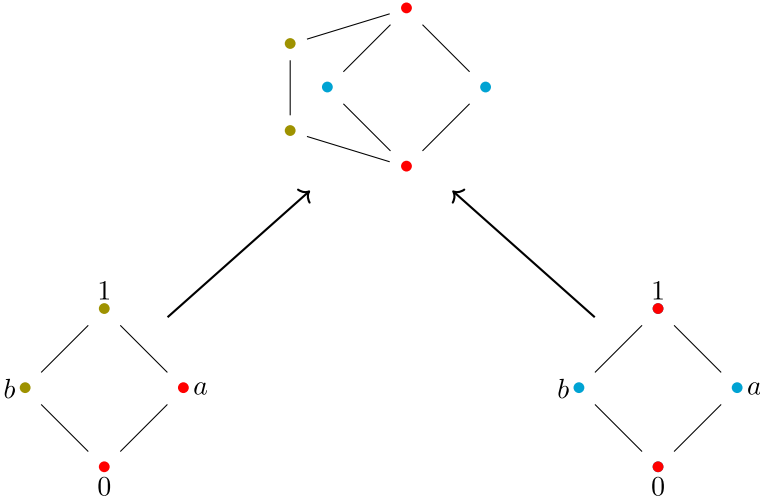


FIGURE 8. Amalgamating the 4-element Boolean lattice with itself

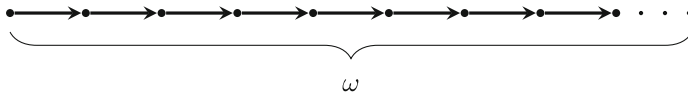
In other words, the network $\mathcal{N}(K)$ of all subgroups of $\mathbb{Q}/\mathbb{Z}$ is a union of infinitely many connected components, each of which is of diameter 0, 1 or 2.

The implication in Proposition 2.12 cannot be replaced by an equivalence. Here is a straightforward example. Suppose that $L = \{0, a, b, 1\}$ is the 4-element Boolean lattice and assume that K is the class of all sublattices of L . Up to isomorphism, there are only 4 sublattices of L , so it is not hard to see that K in this case has the push-up property. We show that K does not have the amalgamation property. We do that by contradiction. Assume that K has the amalgamation property. Consider the sublattice $L' = \{0, 1\}$ and the following two embeddings of L' into L :

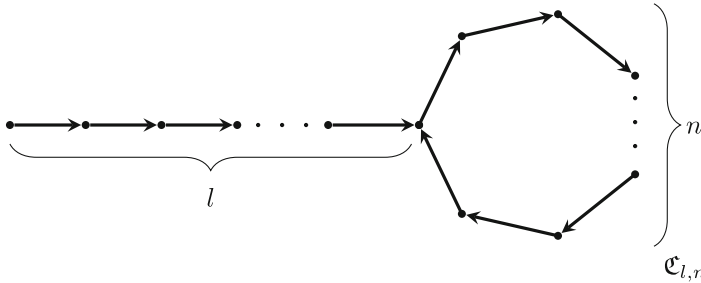
Then, by the amalgamation property, there is a lattice $L'' \subseteq L$ that amalgamates L with itself through L' and the embeddings $f = \{(0, 0), (1, 1)\}$ and $g = \{(0, 0), (1, a)\}$, hence L'' contains at least 6 elements, see Figure 8. This is a contradiction, a subalgebra of a 4-element algebra cannot have more than 4 elements.

3. Networks of monounary algebras

In this section, we give a persuasive example of a network of large embeddings. This is a network of easily visualizable structures of the form $\mathfrak{A} = \langle A, f \rangle$ with one unary operation f . Such a structure is called *monounary algebra*. A nice feature of monounary algebras is that it is easy to describe them pictorially. The algebra $\mathfrak{A} = \langle A, f \rangle$ can be represented by the directed graph (A, E_A) , with nodes A and edges $E_A = \{(a, f(a)) \mid a \in A\}$. For instance, Figure 9 gives pictorial graphical descriptions of some important examples of monounary algebras.



(A) Infinite chain of length ω . This algebra is denoted by $\mathfrak{C}_\omega$.



(B) Finite tail of length l attached to a finite cycle of length $n \neq 0$. This algebra is denoted by $\mathfrak{C}_{l,n}$. In case $l = 0$, we simply write $\mathfrak{C}_n$ in place of $\mathfrak{C}_{0,n}$.

FIGURE 9. Pictorial examples of monounary algebras

The theory of monounary algebras has an extensive literature, and the research in this area is still active, see e.g., [9, 3, 7, 10, 11, 14]. Before we begin our investigation, we list some known facts about these algebras, see, e.g., [21]. Throughout this section, unless stated otherwise, $\mathbf{K}$ is the class of all monounary algebras and $\mathfrak{A} = \langle A, f \rangle$ is an arbitrary monounary algebra.

We say that $\mathfrak{A}$ is *connected* if for any $a, b \in A$ there are $k, m \in \omega$ such that $f^k(a) = f^m(b)$. Let $\mathfrak{B} = \langle B, g \rangle$ be a monounary algebra. Let $\mathfrak{A} \sqcup \mathfrak{B}$ denote the *disjoint union* of the algebras $\mathfrak{A}$ and $\mathfrak{B}$, whose universe $A \sqcup B$ is the disjoint union of A and B , and whose unary operation is the straightforward generalization of the operations f and g to $A \sqcup B$. Finite and infinite disjoint unions of monounary algebras are defined as the natural generalizations of the operation $\sqcup$.

(MU1) Every monounary algebra can be uniquely decomposed into a disjoint union (finite or infinite) of connected monounary algebras.

In other words, the monounary algebra $\mathfrak{A}$ is connected exactly iff its directed graph representation is connected in the sense of graph theory. By (MU1), understanding the class of monounary algebras amounts to understand its connected members. According to [21, Theorem 3.3], we have the following characterization of connected monounary algebras, where by an *in-tree*, we understand a directed graph in which, for a particular vertex r (called the *root*) and any other vertex u , there is exactly one directed path from u to r .

(MU2) If $\mathfrak{A}$ is connected, then $\mathfrak{A}$ has a subalgebra that is isomorphic either to $\mathfrak{C}_\omega$, or to $\mathfrak{C}_n$ for some non-zero $n \in \omega$. This subalgebra is called a *core* of $\mathfrak{A}$, and it is unique up to isomorphism. Deleting the edges of a core from the directed graph representation of $\mathfrak{A}$ would result in a

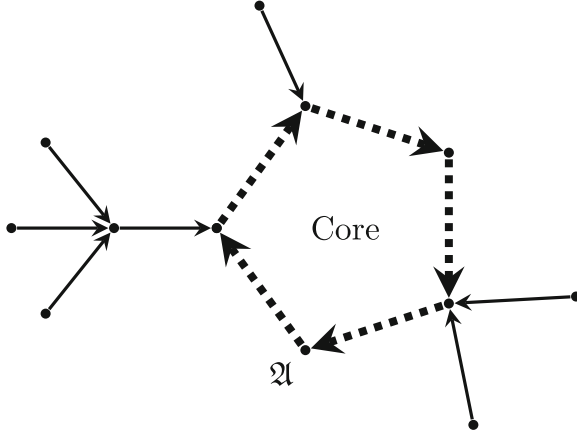


FIGURE 10. Core and in-trees rooted in the vertices of this core

union of disjoint in-trees each of which is rooted in an element of the core. See Figure 10. (Note that, by definition, there are no loops at the roots of in-trees, hence they are not directed graphs corresponding to monounary algebras by themselves, unless we attach some core to their roots.)

When no confusion is likely, we use the phrase “the core” to mean the isomorphism type of the cores. It is worth noting that if a core is finite, then it is unique (not only up to isomorphism).

Speaking of in-trees, we recall some related graph-theoretical notions. In an in-tree, the *parent* of a vertex v is the vertex adjacent to v in the path towards the root; every vertex has a unique parent except the root which has no parent. A *child* of a vertex v is a vertex of which v is the parent. A *leaf* is a vertex that has no children but has a parent (i.e. it is not a root). Now, we will highlight the interconnection between the leaves of the in-trees quoted in (MU2) and the generators of $\mathfrak{A}$.

The monounary algebra $\mathfrak{A}$ is said to have an *infinite tail* if there is an infinite sequence

$$a_0, a_1, a_2, \dots \in A$$

of elements of A such that $a_n \neq a_m$ and $f(a_{n+1}) = a_n$ for each $n, m \in \omega$ with $n \neq m$. An element $a \in A$ is called a *leaf* if there is no $b \in A$ with $f(b) = a$. The set of all leaves of $\mathfrak{A}$ is denoted by $\text{leaves}(\mathfrak{A})$. We denote the *minimum number of generators* of $\mathfrak{A}$ by $d(\mathfrak{A})$. If $\mathfrak{A}$ is not finitely generated, then $d(\mathfrak{A})$ is defined to be ∞ .

(MU3) Assume that $\bigsqcup_{i \in I} \mathfrak{A}_i$ is a disjoint union of connected monounary algebras, for some set I , then

$$d\left(\bigsqcup_{i \in I} \mathfrak{A}_i\right) = \sum_{i \in I} d(\mathfrak{A}_i).$$

So, in particular, if $\mathfrak{A}$ can be decomposed into an infinite disjoint union of connected monounary algebras, then $d(\mathfrak{A}) = \infty$. Now, suppose that $\mathfrak{A}$ is a connected algebra. One can effortlessly find a core for $\mathfrak{A}$ such that $\mathbf{leaves}(\mathfrak{A})$ is the set of all leafs of all in-trees rooted in the vertices of this core, cf. (MU2). Whence, the following are legitimate.

(MU3a) If $\mathfrak{A}$ has an infinite tail, then it cannot be finitely generated, i.e. $d(\mathfrak{A}) = \infty$.

(MU3b) If $\mathfrak{A}$ has no infinite tails and $|\mathbf{leaves}(\mathfrak{A})| \geq 1$, then $d(\mathfrak{A}) = |\mathbf{leaves}(\mathfrak{A})|$.

Here, $|\mathbf{leaves}(\mathfrak{A})|$ is the number of elements in $\mathbf{leaves}(\mathfrak{A})$ if it is finite, and ∞ otherwise.

(MU3c) If $\mathfrak{A}$ has no infinite tails and $\mathbf{leaves}(\mathfrak{A}) = \emptyset$, then $d(\mathfrak{A}) = 1$. In this case, $\mathfrak{A}$ is isomorphic to $\mathfrak{C}_n$ for some non-zero $n \in \omega$.

(MU4) Every monounary algebra generated by a single element is isomorphic to either $\mathfrak{C}_\omega$ or $\mathfrak{C}_{l,n}$ for some $l, n \in \omega$ and $n \neq 0$. See Figure 9.

Proposition 3.1. *The following is true for any monounary algebras $\mathfrak{A}$ and $\mathfrak{B}$:*

$$\mathfrak{A} \text{ is embeddable into } \mathfrak{B} \implies d(\mathfrak{A}) \leq d(\mathfrak{B}).$$

Proof. It is enough to prove the proposition for connected monounary algebras $\mathfrak{A}$ and $\mathfrak{B}$. We can also assume that $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$. If $d(\mathfrak{B}) = \infty$, then there is nothing to prove. So, we can assume that $d(\mathfrak{B}) < \infty$. Thus, neither of $\mathfrak{A}$ and $\mathfrak{B}$ can have any infinite tail; otherwise (MU3a) would contradict the finiteness of $d(\mathfrak{B})$. If $\mathbf{leaves}(\mathfrak{A}) = \emptyset$, then we are obviously done by (MU3c). Now, we assume that $\mathbf{leaves}(\mathfrak{A}) \neq \emptyset$. We will show that $|\mathbf{leaves}(\mathfrak{A})| \leq |\mathbf{leaves}(\mathfrak{B})|$. To this end, we define a function $\psi: \mathbf{leaves}(\mathfrak{A}) \rightarrow \mathbf{leaves}(\mathfrak{B})$ as follows. Let $a \in \mathbf{leaves}(\mathfrak{A})$. If a is a leaf in $\mathfrak{B}$, then we define $\psi(a) = a$. If not, then there is $a_1 \in \mathfrak{B}$ such that $f(a_1) = a$. If a_1 is a leaf in $\mathfrak{B}$, then we define $\psi(a) = a_1$. If not, then there is $a_2 \in \mathfrak{B}$ such that $f(a_2) = a_1$. We can continue in this procedure till we find a leaf $a_n \in \mathfrak{B}$ such that $f^n(a_n) = a$. This procedure will halt because $\mathfrak{B}$ does not contain an infinite tail. Finally, we define $\psi(a) = a_n$. It remains to prove that ψ is one-to-one. Let $a, b \in \mathbf{leaves}(\mathfrak{A})$ be such that $\psi(a) = \psi(b)$. By the construction of ψ , there are $n, m \in \omega$ such that $a = f^n(\psi(a))$ and $b = f^m(\psi(b))$. Without loss of generality, we can assume that $n \leq m$. So, we can find a natural number $k \in \omega$ such that $n + k = m$. Since f is a function, then the following is true in $\mathfrak{B}$:

$$b = f^m(\psi(b)) = f^{n+k}(\psi(b)) = f^k(f^n(\psi(b))) = f^k(f^n(\psi(a))) = f^k(a).$$

The equation $b = f^k(a)$ must be true in the subalgebra $\mathfrak{A}$ too because $\mathfrak{A}$ is closed under the operation f . That implies that $k = 0$, because if $k > 0$, then we will have a contradiction with the fact that both a and b are leaves in $\mathfrak{A}$. Therefore, $a = b$ and ψ is one-to-one as desired. $\square$

Now we are ready to state our main result in this section. For each $n \in \omega$, S_n denotes the set of all permutations on the set $\{i \in \omega \mid i < n\}$. From now on, the symbols i, j, k will be used for indexes that vary among natural numbers in ω .

Theorem 3.2. *Let $n \leq m$ be two positive natural numbers, let $\mathfrak{A}$ and $\mathfrak{B}$ be two monounary algebras having n -many and m -many connected components, and let $\mathfrak{A} = \bigsqcup_{i < n} \mathfrak{A}_i$ and $\mathfrak{B} = \bigsqcup_{j < m} \mathfrak{B}_j$ be their decomposition into disjoint unions of connected monounary algebras. Then*

$$d_g(\mathfrak{A}, \mathfrak{B}) = \min_{\pi \in S_m} \left(\sum_{i < n} d_g(\mathfrak{A}_i, \mathfrak{B}_{\pi(i)}) + \sum_{n \leq k < m} d(\mathfrak{B}_{\pi(k)}) \right). \quad (3.1)$$

Roughly speaking, Theorem 3.2 states that in order to determine the distance between two monounary algebras $\mathfrak{A}$ and $\mathfrak{B}$ having finitely many connected components it is enough to determine the distance between their components, and then the distance of $\mathfrak{A}$ and $\mathfrak{B}$ can be calculated by finding a “best matching” of the components that minimizes the sums in equation (3.1). To prove Theorem 3.2, we need first to prove some key propositions that seem interesting in their own.

The class K of all monounary algebras is closed under the formation of subalgebras. So, in order to characterize the network of large embeddings in K , it is enough to consider the black edges that represent the symmetric closure of the large embedding relation. We start with characterizing the large embeddings between monounary algebras. To do so, we introduce a monounary algebra constructed from another one by attaching an n -long “tail” at some element as follows.

Definition 3.3. Let $\mathfrak{A} = \langle A, f \rangle$ be a monounary algebra, and let $a \in A$ and let $n \in \omega$. We define a monounary algebra $\mathfrak{A} +_a n$ as follows: Let $\{x_i \mid i < n\}$ be a set of size n of brand-new elements none of which is an element of A , and let

$$\mathfrak{A} +_a n \stackrel{\text{def}}{=} \langle A \sqcup \{x_i \mid i < n\}, \hat{f} \rangle$$

where $\hat{f}$ is defined as

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } x \in A, \\ x_{i+1} & \text{if } x = x_i \text{ and } i < n-1, \text{ and} \\ a & \text{if } n \neq 0 \text{ and } x = x_{n-1}. \end{cases}$$

Note that, for all $a, b \in A$ and $n, m \in \omega$, we have

$$\mathfrak{A} +_a 0 \cong \mathfrak{A} \quad \text{and} \quad \mathfrak{A} +_a n +_b m \cong \mathfrak{A} +_b m +_a n. \quad (3.2)$$

Proposition 3.4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two monounary algebras. Then*

$$\begin{aligned} \mathfrak{A} \xrightarrow{\langle \cdot \rangle} \mathfrak{B} \text{ iff one of the following holds:} \\ \mathfrak{B} \cong \mathfrak{A} \sqcup \mathfrak{C}_\omega, \text{ or} \\ \mathfrak{B} \cong \mathfrak{A} \sqcup \mathfrak{C}_{l,n} \text{ for some } l, n \in \omega \text{ and } n \neq 0, \text{ or} \\ \mathfrak{B} \cong \mathfrak{A} +_a m \text{ for some } m \in \omega \text{ and } a \in A. \end{aligned} \quad (3.3)$$

Proof. The direction “ $\Leftarrow$ ” is straightforward by (MU4) and the construction in Definition 3.3. To show the other direction “ $\Rightarrow$ ”, we assume that $\mathfrak{A} \xrightarrow{\langle \cdot \rangle} \mathfrak{B}$,

then there is a subalgebra $\mathfrak{A}' \subseteq \mathfrak{B}$ such that $\mathfrak{A}'$ is isomorphic to $\mathfrak{A}$, and $\mathfrak{B} = \langle A' \cup \{b\} \rangle$ for some $b \in B$. Let g denote the unary operation of $\mathfrak{B}$.

There are two cases:

Case 1: Suppose that $g^n(b) \notin A'$ for every $n \in \omega$. In this case, the subalgebra of $\mathfrak{B}$ generated by b must be disjoint from A' . By (MU4), $\langle b \rangle$ is either isomorphic to $\mathfrak{C}_\omega$ or to $\mathfrak{C}_{l,n}$ for some appropriate natural numbers n and l . Since $\langle A' \rangle \cong \mathfrak{A}$, this gives the first two cases of (3.3).

Case 2: Suppose that $g^m(b) \in A'$ for some $m \in \omega$ such that $g^{m-1}(b) \notin A'$ or $g^0(b) = b \in A'$. Then it is not hard to see that $\mathfrak{B} \cong \mathfrak{A}' +_{a'} m$, where $a' = g^m(b)$. Hence, there is $a \in A$ such that $\mathfrak{B} \cong \mathfrak{A}' +_{a'} m \cong \mathfrak{A} +_a m$, and this corresponds to the third case of (3.3). $\square$

The class of monounary algebras has the amalgamation property, cf. e.g., [18, p. 98] and [2]; hence, by Proposition 2.12, it also has the push-up property. Below, we give a direct proof that gives insights on why $\mathbf{K}$ has the push-up property.

Proposition 3.5. *The class of monounary algebras has the push-up property.*

Proof. We should show that, for all $\mathfrak{A}, \mathfrak{B} \in \mathbf{K}$,

$$(\exists \mathfrak{C} \in \mathbf{K}) \quad \mathfrak{A} \xleftarrow{\langle \bullet \rangle} \mathfrak{C} \xrightarrow{\langle \bullet \rangle} \mathfrak{B} \implies (\exists \mathfrak{D} \in \mathbf{K}) \quad \mathfrak{A} \xrightarrow{\langle \bullet \rangle} \mathfrak{D} \xleftarrow{\langle \bullet \rangle} \mathfrak{B}. \quad (3.4)$$

By Proposition 3.4, there are nine cases to be considered to prove (3.4). For the eight cases when either $\mathfrak{A}$ or $\mathfrak{B}$ can be achieved from $\mathfrak{C}$ by a disjoint union, the desired algebra $\mathfrak{D}$ can be easily constructed using disjoint union, e.g., if $\mathfrak{A} \cong \mathfrak{C} \sqcup \mathfrak{C}_\omega$, then $\mathfrak{D} = \mathfrak{B} \sqcup \mathfrak{C}_\omega$, etc.

Let us now assume that $\mathfrak{A} \cong \mathfrak{C} +_{c_1} n$ and $\mathfrak{B} \cong \mathfrak{C} +_{c_2} m$ for some natural numbers $n, m \in \omega$ and $c_1, c_2 \in \mathfrak{C}$, and let $\alpha: \mathfrak{C} +_{c_1} n \rightarrow \mathfrak{A}$ and $\beta: \mathfrak{C} +_{c_2} m \rightarrow \mathfrak{B}$ be the corresponding isomorphisms. Let $a = \alpha(c_2)$ and $\mathfrak{D} = \mathfrak{A} +_a m$. Then, by (3.2), it follows that $\mathfrak{D} \cong \mathfrak{B} +_b n$ where $b = \beta(c_1)$. $\square$

Let us note that the converse of (3.4) fails in the network $\mathcal{N}(\mathbf{K})$ of monounary algebras because, for example, if $\mathfrak{A} = \mathfrak{C}_\omega$ and $\mathfrak{B} = \mathfrak{C}_{l,n}$ for some $n > 0$ and l , then there is no $\mathfrak{C}$ which is embeddable into both $\mathfrak{A}$ and $\mathfrak{B}$, but both $\mathfrak{A}$ and $\mathfrak{B}$ are largely embeddable into $\mathfrak{D} = \mathfrak{C}_\omega \sqcup \mathfrak{C}_{l,n}$. However, if we restrict the class $\mathbf{K}$ to be the class of connected monounary algebras, then we have both directions of (3.4). In other words, the class of all monounary algebras has the push-up property but lacks the push-down property, while the class of connected monounary algebras has both properties.

Proposition 3.6. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two connected monounary algebras. If $\mathfrak{A}$ and $\mathfrak{B}$ have isomorphic cores, then*

$$d_g(\mathfrak{A}, \mathfrak{B}) \leq d(\mathfrak{A}) + d(\mathfrak{B}) - 1. \quad (3.5)$$

If the cores of $\mathfrak{A}$ and $\mathfrak{B}$ are not isomorphic, then

$$d_g(\mathfrak{A}, \mathfrak{B}) = d(\mathfrak{A}) + d(\mathfrak{B}). \quad (3.6)$$

Proof. In general, it is true that

$$d_g(\mathfrak{A}, \mathfrak{B}) \leq d(\mathfrak{A}) + d(\mathfrak{B}) \quad (3.7)$$

because $\mathfrak{B} \xrightarrow{(\bullet)}_{\times d(\mathfrak{A})} \mathfrak{A} \sqcup \mathfrak{B}$ and $\mathfrak{A} \xrightarrow{(\bullet)}_{\times d(\mathfrak{B})} \mathfrak{A} \sqcup \mathfrak{B}$ if both $\mathfrak{A}$ and $\mathfrak{B}$ are finitely generated, and this inequality reduces to the trivial $d_g(\mathfrak{A}, \mathfrak{B}) \leq \infty$ otherwise.

Assume first that the cores of $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic. Now, we need to prove (3.5). If one of the algebras $\mathfrak{A}$ and $\mathfrak{B}$ is infinitely generated, then there is nothing to prove. Assume that both $\mathfrak{A}$ and $\mathfrak{B}$ are finitely generated. We are going to prove (3.5) by induction on $k = d(\mathfrak{A}) + d(\mathfrak{B})$.

If $k = 2$, then either $\mathfrak{A} \cong \mathfrak{B} \cong \mathfrak{C}_\omega$ or $\mathfrak{A} \cong \mathfrak{C}_{p,n}$ and $\mathfrak{B} \cong \mathfrak{C}_{q,n}$ for some natural numbers p, q and n . So, we have $d_g(\mathfrak{A}, \mathfrak{B}) \leq 1$ in both cases. Inductively, assume that (3.5) is true for any two connected monounary algebras $\mathfrak{A}'$ and $\mathfrak{B}'$ having isomorphic cores and satisfying $d(\mathfrak{A}') + d(\mathfrak{B}') = N$ for some natural $N \geq 2$. Now, suppose that $d(\mathfrak{A}) + d(\mathfrak{B}) = N + 1$. Then at least one of the two algebras, say $\mathfrak{A}$, has more than one generator. Thus, one can find a subalgebra $\mathfrak{A}'$ of $\mathfrak{A}$ such that $d(\mathfrak{A}') = d(\mathfrak{A}) - 1$ and $\mathfrak{A}' \xrightarrow{(\bullet)} \mathfrak{A}$. Hence, by the induction hypothesis, we have $d_g(\mathfrak{A}', \mathfrak{B}) \leq d(\mathfrak{A}') + d(\mathfrak{B}) - 1$. Therefore,

$$d_g(\mathfrak{A}, \mathfrak{B}) \leq d_g(\mathfrak{A}, \mathfrak{A}') + d_g(\mathfrak{A}', \mathfrak{B}) \leq d(\mathfrak{A}') + d(\mathfrak{B}) = d(\mathfrak{A}) + d(\mathfrak{B}) - 1,$$

and this is what we wanted to show.

Assume now that the cores of $\mathfrak{A}$ and $\mathfrak{B}$ are not isomorphic. To prove equation (3.6), by (3.7), it is enough to show that $d_g(\mathfrak{A}, \mathfrak{B}) \geq d(\mathfrak{A}) + d(\mathfrak{B})$. If $d_g(\mathfrak{A}, \mathfrak{B}) = \infty$, then we have nothing to prove. So, let us assume that $d_g(\mathfrak{A}, \mathfrak{B}) < \infty$. Then, by Proposition 3.5 and Proposition 2.11, there is a monounary algebra $\mathfrak{D}$ and $n, m \in \omega$ such that $d_g(\mathfrak{A}, \mathfrak{B}) = n + m$, $\mathfrak{A} \xrightarrow{(\bullet)}_{\times m} \mathfrak{D}$ and $\mathfrak{B} \xrightarrow{(\bullet)}_{\times n} \mathfrak{D}$. Hence, $\mathfrak{D}$ contains isomorphic images of both $\mathfrak{A}$ and $\mathfrak{B}$. Since the cores of $\mathfrak{A}$ and $\mathfrak{B}$ are not isomorphic, these images should be contained in different components of $\mathfrak{D}$. Therefore, $\mathfrak{D}$ must also contain an isomorphic image of $\mathfrak{A} \sqcup \mathfrak{B}$. Thus, $n \geq d(\mathfrak{A})$ because by Proposition 3.1 one have to add at least an isomorphic copy of $\mathfrak{A}$ to $\mathfrak{B}$ in order to generate $\mathfrak{D}$. Similarly, we must have $m \geq d(\mathfrak{B})$. Consequently, $d_g(\mathfrak{A}, \mathfrak{B}) = n + m \geq d(\mathfrak{A}) + d(\mathfrak{B})$, and we are done. $\square$

Now, we will prove a formula that gives the generator distance between two monounary algebras (with finitely many connected components) in terms of the distance between their components. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two monounary algebras, each of which has finitely many connected components. Assume that $d_g(\mathfrak{A}, \mathfrak{B})$ is finite, then by Proposition 2.11 there are $p, q \in \omega$ and $\mathfrak{D} \in K$ such that

$$\mathfrak{A} \xrightarrow{(\bullet)}_{\times p} \mathfrak{D}, \quad \mathfrak{B} \xrightarrow{(\bullet)}_{\times q} \mathfrak{D} \quad \text{and} \quad d_g(\mathfrak{A}, \mathfrak{B}) = p + q.$$

Let $\mathfrak{A} - \mathfrak{D} - \mathfrak{B}$ denote the corresponding path from $\mathfrak{A}$ to $\mathfrak{B}$ through the algebra $\mathfrak{D}$. Let $\alpha: \mathfrak{A} \rightarrow \mathfrak{D}$ and $\beta: \mathfrak{B} \rightarrow \mathfrak{D}$ be the respective compositions of large embeddings given by relations $\mathfrak{A} \xrightarrow{(\bullet)}_{\times p} \mathfrak{D}$ and $\mathfrak{B} \xrightarrow{(\bullet)}_{\times q} \mathfrak{D}$. We will refer to the condition $d_g(\mathfrak{A}, \mathfrak{B}) = p + q$ as the *minimality condition*.

Note that two elements that belong to the same connected component of a monounary algebra must have their images under any homomorphism

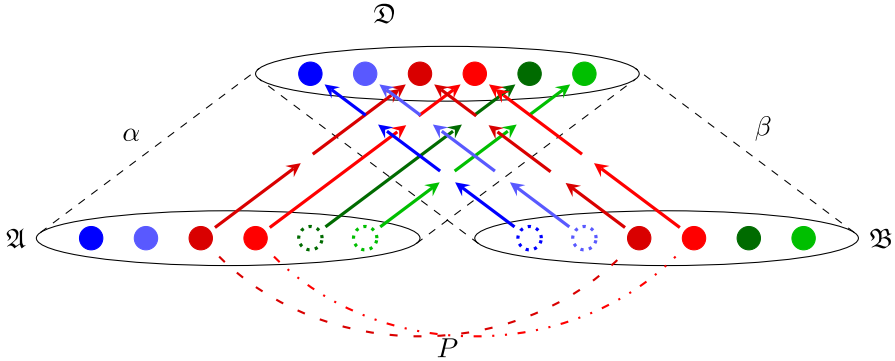


FIGURE 11. Coloring the components of the algebras $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{D}$, and the large embedding steps in the path $\mathfrak{A} - \mathfrak{D} - \mathfrak{B}$

again in the same connected component. Hence, each component of $\mathfrak{A}$ and $\mathfrak{B}$ is mapped into a connected component of $\mathfrak{D}$ by α and β . We also have that α maps the different components of $\mathfrak{A}$ into different components of $\mathfrak{D}$ because α is one-to-one, and the core of a component of $\mathfrak{A}$ has to be mapped into the core of a component of $\mathfrak{D}$. The same holds for β and the components of $\mathfrak{B}$ for the same reasons.

Thus, there are injective functions $\Psi_{\mathfrak{A}}$ and $\Psi_{\mathfrak{B}}$ from the components of $\mathfrak{A}$ and the components of $\mathfrak{B}$, respectively, into the components of $\mathfrak{D}$. For easy reference and visualization, we use a coloring style that codes these injective functions as follows.

- We color each component of $\mathfrak{D}$ that appeared in both the range of $\Psi_{\mathfrak{A}}$ and the range of $\Psi_{\mathfrak{B}}$ with a different shade of red.
- We color each component of $\mathfrak{D}$ that appeared in the range of $\Psi_{\mathfrak{A}}$ but not in the range of $\Psi_{\mathfrak{B}}$ with a different shade of blue.
- We color each component of $\mathfrak{D}$ that appeared in the range of $\Psi_{\mathfrak{B}}$ but not in the range of $\Psi_{\mathfrak{A}}$ with a different shade of green.
- We color each component of $\mathfrak{D}$ that appears neither in the range of $\Psi_{\mathfrak{B}}$ nor in the range of $\Psi_{\mathfrak{A}}$ with a different shade of gray.

By Proposition 3.4, we know that each large embedding in $\mathfrak{A} \xrightarrow{(\bullet)}_{\times_p} \mathfrak{D}$ adds a piece of a component of the big algebra $\mathfrak{D}$ to the smaller algebra $\mathfrak{A}$; it adds either a new core or a tail to an existing component. The same also is true for each large embedding in $\mathfrak{B} \xrightarrow{(\bullet)}_{\times_q} \mathfrak{D}$.

- We color each large embedding step in $\mathfrak{A} - \mathfrak{D} - \mathfrak{B}$ with the same color of the component of $\mathfrak{D}$ to which this large embedding is contributing in its construction.
- We color each component of $\mathfrak{A}$ and $\mathfrak{B}$ with the same color of its image under the injective functions $\Psi_{\mathfrak{A}}$ and $\Psi_{\mathfrak{B}}$, respectively.

Note that there is no green component of $\mathfrak{A}$, and there is no blue component of $\mathfrak{B}$. It also turned out that there cannot be a gray component of $\mathfrak{D}$ and/or a gray large embedding in $\mathfrak{A} - \mathfrak{D} - \mathfrak{B}$, see Figure 11. This is true because

by deleting all the gray components and the gray large embeddings we would get a shorter path; which is a contradiction. Note that a large embedding step of some color affects only the component of $\mathfrak{D}$ that has the same color. This fact will be used quite frequently, so we give it a name. We call this feature the *independence of the colors*.

Let R , B and G be the sets of all shades of red, blue and green, respectively, that were used in the above coloring. The component of $\mathfrak{A}$ which is colored by color c is denoted by $\mathfrak{A}_c$. The same applies to the components of the monounary algebras $\mathfrak{B}$ and $\mathfrak{D}$.

Let $b \in B$ be a blue color. The total number of the b -colored large embedding steps in $\mathfrak{B} \xrightarrow{(\bullet)}_{\times q} \mathfrak{D}$ gives the minimum number of generators of $\mathfrak{D}_b$; this is the shortest possible way to build $\mathfrak{D}_b$ from scratch. If not, then one can use the independence of colors to derive a contradiction with the minimality condition. Similar conclusions can be drawn regarding the green components of $\mathfrak{D}$ and the green large embeddings in $\mathfrak{A} \xrightarrow{(\bullet)}_{\times p} \mathfrak{D}$.

If $\mathfrak{A}_b$ is not isomorphic to $\mathfrak{D}_b$, then we can use Proposition 3.1 and the independence of colors to shorten the path $\mathfrak{A} - \mathfrak{D} - \mathfrak{B}$ by replacing the component $\mathfrak{D}_b$ with a smaller one isomorphic to $\mathfrak{A}_b$ in the big algebra $\mathfrak{D}$, and this would contradict the minimality condition. Thus, there is no b -colored large embedding in $\mathfrak{A} \xrightarrow{(\bullet)}_{\times p} \mathfrak{D}$. The same holds for the green components of $\mathfrak{B}$ and $\mathfrak{D}$. Hence,

(A) For each $b \in B$ and each $g \in G$, we have

the number of all b -colored large embedding steps in $\mathfrak{A} - \mathfrak{D} - \mathfrak{B} = d(\mathfrak{D}_b)$,
the number of all g -colored large embedding steps in $\mathfrak{A} - \mathfrak{D} - \mathfrak{B} = d(\mathfrak{D}_g)$.

(B) For each $b \in B$ and each $g \in G$, we have

$$\mathfrak{A}_b \cong \mathfrak{D}_b \quad \& \quad \mathfrak{B}_g \cong \mathfrak{D}_g.$$

Let $b \in B$ and $g \in G$. All the b -colored and the g -colored large embeddings in $\mathfrak{A} - \mathfrak{D} - \mathfrak{B}$ form a path between $\mathfrak{A}_b$ and $\mathfrak{B}_g$. This can be guaranteed by the independence of colors. By (A) and (B), the length of this path is $d(\mathfrak{A}_b) + d(\mathfrak{B}_g)$. On the other hand, this path should have the shortest length among all paths connecting $\mathfrak{A}_b$ and $\mathfrak{B}_g$ in $\mathcal{N}(K)$. Otherwise, by the independence of colors, one can construct a path between $\mathfrak{A}$ and $\mathfrak{B}$ that is shorter than $\mathfrak{A} - \mathfrak{D} - \mathfrak{B}$. Thus, Proposition 3.6 implies the observation (C) below.

(C) For each $b \in B$ and each $g \in G$, $\mathfrak{A}_b$ and $\mathfrak{B}_g$ must have none isomorphic cores, and

$$d_g(\mathfrak{A}_b, \mathfrak{B}_g) = d(\mathfrak{A}_b) + d(\mathfrak{B}_g).$$

It is time now to talk about the red components. Again, the next observation follows by the minimality condition and the independence of colors:

(D) For each $r \in R$, the cores of $\mathfrak{A}_r$, $\mathfrak{B}_r$ and $\mathfrak{D}_r$ must be isomorphic, and the number of all r -colored large embedding steps in $\mathfrak{A} - \mathfrak{D} - \mathfrak{B}$ equals $d_g(\mathfrak{A}_r, \mathfrak{B}_r)$.

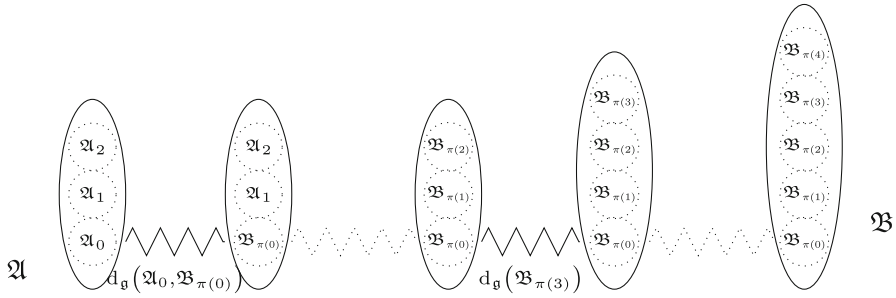


FIGURE 12. The path $\mathfrak{A} - \pi - \mathfrak{B}$ between $\mathfrak{A}$ and $\mathfrak{B}$ based on the permutation π

Therefore, by (A), (B) and (C), we have

$$d_g(\mathfrak{A}, \mathfrak{B}) = \sum_{b \in B} d(\mathfrak{A}_b) + \sum_{g \in G} d(\mathfrak{B}_g) + \sum_{r \in R} d_g(\mathfrak{A}_r, \mathfrak{B}_r). \quad (3.8)$$

Now, we are ready to prove our main Theorem.

Proof of Theorem 3.2. We recall the assumptions of the theorem. Let $n \leq m$ be two natural numbers, let $\mathfrak{A}$ and $\mathfrak{B}$ be two monounary algebras having n -many and m -many connected components, and let $\mathfrak{A} = \bigsqcup_{i < n} \mathfrak{A}_i$ and $\mathfrak{B} = \bigsqcup_{j < m} \mathfrak{B}_j$ be their decomposition into disjoint unions of connected monounary algebras. We need to prove that

$$d_g(\mathfrak{A}, \mathfrak{B}) = \min_{\pi \in S_m} \left(\sum_{i < n} d_g(\mathfrak{A}_i, \mathfrak{B}_{\pi(i)}) + \sum_{n \leq k < m} d(\mathfrak{B}_{\pi(k)}) \right). \quad (3.9)$$

It is easy to see that

$$d_g(\mathfrak{A}, \mathfrak{B}) \leq \sum_{i < n} d_g(\mathfrak{A}_i, \mathfrak{B}_{\pi(i)}) + \sum_{n \leq k < m} d(\mathfrak{B}_{\pi(k)}) \quad \text{for all } \pi \in S_m.$$

Indeed, for $\pi \in S_m$, the algebra $\mathfrak{B}$ can be reached from $\mathfrak{A}$ by joining the $d_g(\mathfrak{A}_i, \mathfrak{B}_{\pi(i)})$ -long paths going from $\mathfrak{A}_i$ to $\mathfrak{B}_{\pi(i)}$, for each $i < n$, and then continuing by building the remaining components of $\mathfrak{B}$ starting from scratch in $\sum_{n \leq k < m} d(\mathfrak{B}_{\pi(k)})$ -many steps, see Figure 12; let $\mathfrak{A} - \pi - \mathfrak{B}$ denote this path between $\mathfrak{A}$ and $\mathfrak{B}$ which is based on the permutation π .

So we have that $d_g(\mathfrak{A}, \mathfrak{B})$ is less than or equal to the minimum in the right-hand side of (3.9). Consequently, if $d_g(\mathfrak{A}, \mathfrak{B}) = \infty$, we have nothing to prove. Let us assume that $d_g(\mathfrak{A}, \mathfrak{B}) < \infty$. By Proposition 2.11, there is a monounary algebra $\mathfrak{D}$ and $p, q \in \omega$ such that

$$\mathfrak{A} \xrightarrow{(\bullet)}_{\times p} \mathfrak{D}, \quad \mathfrak{B} \xrightarrow{(\bullet)}_{\times q} \mathfrak{D} \quad \text{and} \quad d_g(\mathfrak{A}, \mathfrak{B}) = p + q.$$

Now, we adopt the coloring system given in Figure 11 and the related discussion and terminology. We have two types of indexes for the components of $\mathfrak{A}$ and $\mathfrak{B}$; namely indexes induced by the component decomposition and indexes induced

by the coloring. We define a correspondence P between certain indexes $i < n$ of components of $\mathfrak{A}$ to that of $\mathfrak{B}$ as follows. The domain of P is the set of all indexes of the red components of $\mathfrak{A}$. Let $\mathfrak{A}_i$ be a red component of $\mathfrak{A}$, then we define $P(i) = j$ where j is the index of the component of $\mathfrak{B}$ that have exactly the same color as $\mathfrak{A}_i$. See Figure 11. Pick a permutation $\pi \in S_m$ that extends the correspondence P . Such a permutation exists because $n \leq m$. For each $i < n \leq k < m$, we have

$\mathfrak{A}_i$ is red iff $\mathfrak{B}_{\pi(i)}$ is red, $\mathfrak{A}_i$ is blue iff $\mathfrak{B}_{\pi(i)}$ is green, and $\mathfrak{B}_{\pi(k)}$ is green.

Now, by (3.8) and (C), we have

$$\begin{aligned}
 & \sum_{i < n} d_g(\mathfrak{A}_i, \mathfrak{B}_{\pi(i)}) + \sum_{n \leq k < m} d(\mathfrak{B}_{\pi(k)}) \\
 &= \sum_{\substack{i < n \\ \mathfrak{A}_i \text{ is blue}}} d_g(\mathfrak{A}_i, \mathfrak{B}_{\pi(i)}) + \sum_{\substack{i < n \\ \mathfrak{A}_i \text{ is red}}} d_g(\mathfrak{A}_i, \mathfrak{B}_{\pi(i)}) + \sum_{n \leq k < m} d(\mathfrak{B}_{\pi(k)}) \\
 &= \sum_{\substack{i < n \\ \mathfrak{A}_i \text{ is blue}}} d(\mathfrak{A}_i) + \sum_{\substack{i < n \\ \mathfrak{A}_i \text{ is blue}}} d(\mathfrak{B}_{\pi(i)}) + \sum_{n \leq k < m} d(\mathfrak{B}_{\pi(k)}) \\
 &\quad + \sum_{\substack{i < n \\ \mathfrak{A}_i \text{ is red}}} d_g(\mathfrak{A}_i, \mathfrak{B}_{\pi(i)}) \\
 &= \sum_{\substack{i < n \\ \mathfrak{A}_i \text{ is blue}}} d(\mathfrak{A}_i) + \sum_{\substack{j < m \\ \mathfrak{B}_j \text{ is green}}} d(\mathfrak{B}_j) + \sum_{\substack{i < n \\ \mathfrak{A}_i \text{ is red}}} d_g(\mathfrak{A}_i, \mathfrak{B}_{\pi(i)}) \\
 &= \sum_{b \in B} d(\mathfrak{A}_b) + \sum_{g \in G} d(\mathfrak{B}_g) + \sum_{r \in R} d_g(\mathfrak{A}_r, \mathfrak{B}_r) \\
 &= d_g(\mathfrak{A}, \mathfrak{B}).
 \end{aligned}$$

Therefore, $d_g(\mathfrak{A}, \mathfrak{B})$ is bigger than or equal to the right-hand side of (3.9), and we are done. $\square$

Theorem 3.2 deals with the case when the two algebras have finitely many components. In fact, it is not too difficult to see that for every two monounary algebras $\mathfrak{A}$ and $\mathfrak{B}$, either $d_g(\mathfrak{A}, \mathfrak{B}) = \infty$ or $\mathfrak{A}$ and $\mathfrak{B}$ differ only in finitely many connected components. This can be proved using the push-up property together with the fact that only one component can be affected by one step of large embedding, see Proposition 3.5 and Proposition 3.4. When $\mathfrak{A}$ and $\mathfrak{B}$ differ only in finitely many connected components, Theorem 3.2 can be used to reduce the distance between these algebras to the distance between their connected components.

Now, we can concentrate on the problem of determining the distance between connected monounary algebras. We already have the distance in the case when the algebras have non-isomorphic cores by Proposition 3.6. Let us consider the case of monounary algebras that have cores isomorphic to $\mathfrak{C}_n$ for some non-zero $n \in \omega$. We are going to have a reduction step similar to what we have made in Theorem 3.2. To do so, we introduce the notion of tree-algebra decomposition.

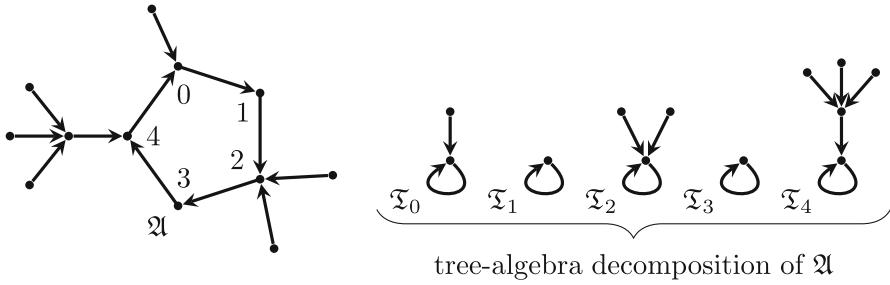


FIGURE 13. Tree-algebra decomposition of a monounary algebra with finite core

Definition 3.7. By a *tree-algebra*, we mean a connected monounary algebra whose core is isomorphic to $\mathfrak{C}_1$. By a *forest-algebra*, we mean a disjoint union of one or more tree-algebras.

Let $\mathfrak{A} = \langle A, f \rangle$ be a connected monounary algebra, and assume that its core is isomorphic to $\mathfrak{C}_n$ for some non-zero $n \in \omega$. Let $\langle T_i \mid i < n \rangle$ be an enumeration of the in-trees whose roots $\langle r_i \mid i < n \rangle$ are elements of $\mathfrak{C}_n$ as described in (MU2). By a *tree-algebra decomposition* of $\mathfrak{A}$, we understand the sequence $\langle \mathfrak{T}_i \mid i < n \rangle$ of tree-algebras, where $\mathfrak{T}_i = \langle T_i, f_i \rangle$ is the tree-algebra corresponding to in-tree T_i such that $f_i(r_i) = r_i$ and f_i maps the other elements of T_i to their parents. See Figure 13 for an illustrative example.

Proposition 3.8. Let $\mathfrak{A} = \langle A, f \rangle$ and $\mathfrak{B} = \langle B, g \rangle$ be two connected monounary algebras having cores isomorphic to $\mathfrak{C}_n$, for some non-zero $n \in \omega$. Let $\langle \mathfrak{T}_i^A \mid i < n \rangle$ and $\langle \mathfrak{T}_i^B \mid i < n \rangle$ be the tree-algebra decompositions of $\mathfrak{A}$ and $\mathfrak{B}$, respectively. Then

$$d_g(\mathfrak{A}, \mathfrak{B}) = \min_{k < n} \sum_{\substack{i, j < n \\ j \equiv i+k \pmod n}} d_g(\mathfrak{T}_i^A, \mathfrak{T}_j^B).$$

Proof. The proof goes exactly in the same spirit as the one of Theorem 3.2; the main differences are that here we do not have unmatched pairs because $\mathfrak{A}$ and $\mathfrak{B}$ have the same number of components in their tree-algebra decompositions since they have isomorphic cores, and that here permutations matching tree-algebras in the two tree-algebra decomposition cannot be arbitrary because they also have to preserve the structure of the common finite core. $\square$

Thus, the distance between connected monounary algebras having isomorphic finite cores can be reduced to the distances between the tree-algebras in their tree-algebra decompositions. Note that Proposition 3.8 does not work for $\mathfrak{C}_\omega$ among others because monounary algebras of core $\mathfrak{C}_\omega$ cannot be uniquely (up to permutation) decomposed into tree-algebras.

So far we have seen how to reduce the distance between monounary algebras of finitely many components to the distances between their connected components and how to reduce the distance between connected algebras of

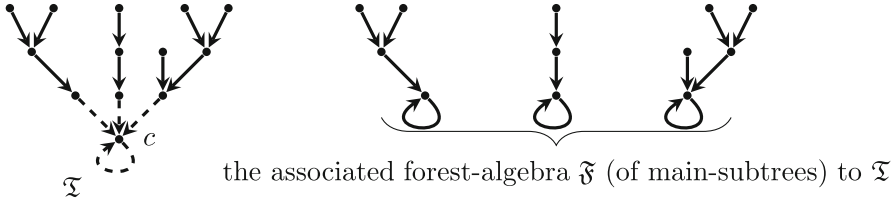


FIGURE 14. A finite tree-algebra broken down into shorter trees

finite cores to the distance between tree-algebras. Now we are going to reduce the distance between finite tree-algebras to the distance between shorter trees.

The idea roughly is to break a tree-algebra $\mathfrak{T}$ down into the main-subtrees rooted in the children of the root of $\mathfrak{T}$. Formally, we do that by constructing a forest-algebra $\mathfrak{F}$ as follows. We remove the core element c together with the arrows arriving to c from the tree $\mathfrak{T}$ and then we attach a loop to every element that was a child of c in the tree-algebra, see Figure 14 for an example illustration. The algebra $\mathfrak{F}$ is called the *associated forest-algebra (of main-subtrees)* to $\mathfrak{T}$.

Proposition 3.9. *Let $\mathfrak{T}_1$ and $\mathfrak{T}_2$ be two tree-algebras, and let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be their associated forest-algebras. Then, $d_g(\mathfrak{T}_1, \mathfrak{T}_2) = d_g(\mathfrak{F}_1, \mathfrak{F}_2)$.*

Consider two finite tree-algebras $\mathfrak{T}_1$ and $\mathfrak{T}_2$. By Proposition 3.9, we know that the distance between $\mathfrak{T}_1$ and $\mathfrak{T}_2$ is the same as the distance between their respected associated algebras of main-subtrees $\mathfrak{F}_1$ and $\mathfrak{F}_2$. Now we can use Theorem 3.2 to reduce the problem to finding distances between pairs of shorter trees, namely the tree-algebras in the tree-algebra decomposition of $\mathfrak{F}_1$ and $\mathfrak{F}_2$. So, all what remains is to prove Proposition 3.9.

Before proving Proposition 3.9, we make some observations concerning forest-algebras in general. Suppose that $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are two forest-algebras of finite distance from each other. Then, by the push-up property, there are $p, q \in \omega$ and $\mathfrak{D} \in K$ such that

$$\mathfrak{F}_1 \xrightarrow{(\bullet)}_{\times p} \mathfrak{D}, \quad \mathfrak{F}_2 \xrightarrow{(\bullet)}_{\times q} \mathfrak{D} \quad \text{and} \quad d_g(\mathfrak{F}_1, \mathfrak{F}_2) = p + q.$$

One can apply the coloring (and the related discussion) of Figure 11 for $\mathfrak{F}_1$ and $\mathfrak{F}_2$ in the place of $\mathfrak{A}$ and $\mathfrak{B}$, respectively. Hence, by (B), (C) and (D), every component of the big algebra $\mathfrak{D}$ must have a core isomorphic to $\mathfrak{C}_1$. Since every algebra in the path $\mathfrak{F}_1 - \mathfrak{D} - \mathfrak{F}_2$ is embeddable into $\mathfrak{D}$, it follows that each component of each algebra in the path $\mathfrak{F}_1 - \mathfrak{D} - \mathfrak{F}_2$ is a tree-algebra.

(MF) Two forest-algebras are of finite distance iff they are connected in the network $\mathcal{N}(K)$ by a minimal path that contains only forest-algebras.

With a very similar argument, we can conclude the same for tree-algebras.

(MT) Two tree-algebras are of finite distance iff they are connected in the network $\mathcal{N}(K)$ by a minimal path that contains only tree-algebras.

Actually we can say more: Two connected monounary algebras $\mathfrak{A}$ and $\mathfrak{B}$ with isomorphic cores are of finite distance iff these algebras are connected in

the network $\mathcal{N}(\mathbf{K})$ by a minimal path that contains only connected algebras with cores isomorphic to the cores of $\mathfrak{A}$ and $\mathfrak{B}$.

Let F be the map taking a tree-algebra to its associated forest-algebra. It is not hard to see that, for every forest-algebra $\mathfrak{F}$, one can find a tree algebra $\mathfrak{T}$ such that $F(\mathfrak{T}) = \mathfrak{F}$. This tree algebra is unique up to isomorphism. We pick one tree-algebra with this property and we denote it by $T(\mathfrak{F})$. The tree-algebra $T(\mathfrak{F})$ can be constructed by deleting all the loops from the tree-algebras in $\mathfrak{F}$ and joining all the roots of these trees to a brand new common root that has a loop. See Figure 14 “backwards” for an illustration. Therefore,

$$F(T(\mathfrak{F})) \cong \mathfrak{F} \quad \text{and} \quad T(F(\mathfrak{T})) \cong \mathfrak{T} \quad (3.10)$$

for all forest-algebra $\mathfrak{F}$ and tree-algebra $\mathfrak{T}$. By their definitions, it is clear that both F and T map isomorphic algebras to isomorphic algebras. Moreover, by Proposition 3.4, we have

$$\begin{aligned} \mathfrak{F} \xrightarrow{(\cdot)} \mathfrak{F}' &\iff T(\mathfrak{F}) \xrightarrow{(\cdot)} T(\mathfrak{F}') \quad \text{and} \\ \mathfrak{T} \xrightarrow{(\cdot)} \mathfrak{T}' &\iff F(\mathfrak{T}) \xrightarrow{(\cdot)} F(\mathfrak{T}'). \end{aligned} \quad (3.11)$$

Now we are ready to prove Proposition 3.9.

Proof of Proposition 3.9. By (MT), a minimal path between $\mathfrak{T}_1$ and $\mathfrak{T}_2$ realizing $d_g(\mathfrak{T}_1, \mathfrak{T}_2)$ contains only tree-algebras. So, by (3.11), taking the F -image of all of the tree-algebras in this minimal path would give a path of the same length between forest-algebras $\mathfrak{F}_1 = F(\mathfrak{T}_1)$ and $\mathfrak{F}_2 = F(\mathfrak{T}_2)$. Hence,

$$d_g(\mathfrak{F}_1, \mathfrak{F}_2) \leq d_g(\mathfrak{T}_1, \mathfrak{T}_2).$$

Analogously, by (MF), a minimal path between $\mathfrak{F}_1$ and $\mathfrak{F}_2$ contains only forest-algebras. So, by (3.10) and (3.11), taking the T -images of the forest-algebras in this path would give a path of the same length between the tree-algebras $\mathfrak{T}_1 \cong T(\mathfrak{F}_1)$ and $\mathfrak{T}_2 \cong T(\mathfrak{F}_2)$. Hence,

$$d_g(\mathfrak{T}_1, \mathfrak{T}_2) \leq d_g(\mathfrak{F}_1, \mathfrak{F}_2).$$

Consequently, $d_g(\mathfrak{T}_1, \mathfrak{T}_2) = d_g(\mathfrak{F}_1, \mathfrak{F}_2)$, and this is what we wanted to show. $\square$

Now we give a concrete example. Consider algebras $\mathfrak{A}$ and $\mathfrak{B}$ illustrated in Figure 15.

There are three main-subtrees of both algebras $\mathfrak{A}$ and $\mathfrak{B}$ of the figure. To calculate the distance between $\mathfrak{A}$ and $\mathfrak{B}$, it is enough to calculate the sum of the distances between these main-subtrees for all the six permutations mapping these subtrees to each other and take the minimum. This follows from Proposition 3.9 and Theorem 3.2. To calculate the distances between any pair of the main-subtrees, we can again use Proposition 3.9 and Theorem 3.2 in the same way. However, since these main-subtrees are so simple, one can easily determine their distances. Therefore, in the figure, we just give their distances at the corresponding permutation without further explanation. Since 4 is the smallest sum, we have $d_g(\mathfrak{A}, \mathfrak{B}) = 4$.

The above considerations give a method by which one can calculate the distance between any two finite monounary algebras. It is clear however that

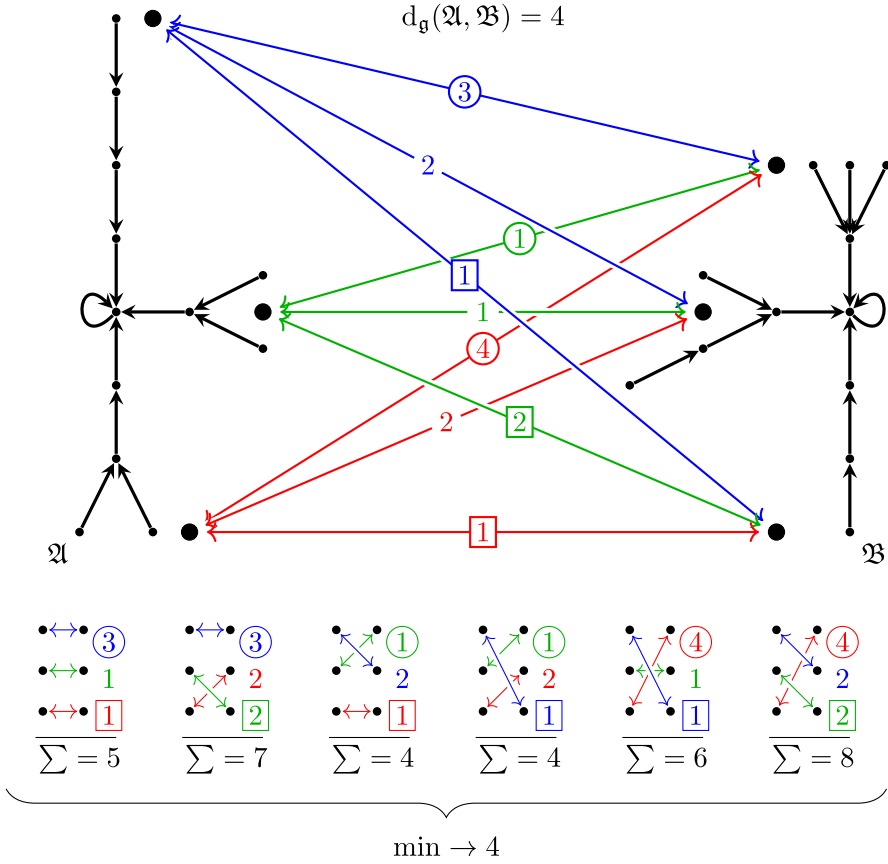


FIGURE 15. The distance between two finite tree-algebras

this method is not very effective because one have to consider lots of permutations in every reduction step. It is a task for further research to find more effective algorithms and to investigate the computational complexity of the problem of determining the generator distance between two finite monounary algebras.

Problem 1. What is the computational complexity of calculating the distance between two finite monounary algebras?

Even though, Proposition 3.9 does not require the tree-algebras to be finite, it does not seem to be of a great help in the case when one of them is infinite. It is not hard though to prove that two arbitrary tree-algebras are of finite distance iff they differ only in finitely many branches, i.e. if both can be embedded into a tree-algebra that differs from each of them only by finitely many extra elements. But even in this case, we do not have a method to calculate this distance yet. We also remind the reader that determining the

distance between two connected monounary algebras, each of which has a core isomorphic to $\mathfrak{C}_\omega$, is still open.

Problem 2. Determine the distance between countable monounary algebras.

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Declarations

Conflict of interest There are no Conflict of interest.

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References

- [1] Beaumont, R.A., Zuckerman, H.S.: A characterization of the subgroups of the additive rationals. *Pacific J. Math.* **1**, 169–177 (1951)
- [2] Berthiaume, P.: The injective envelope of S-sets. *Canad. Math. Bull.* **10**, 261–273 (1967)
- [3] Černegová, M., Jakubíková-Studenovská, D.: The lattice of convexities of partial monounary algebras. *Algebra Univ.* **79**, 10 (2018)
- [4] Clifford, A.H., Preston, G.B.: *The Algebraic Theory of Semigroups*, vol. II. American Mathematical Soc, Providence (1967)
- [5] Dwinger, Ph., Yaqub, F.M.: Generalized free products of Boolean algebras with an amalgamated subalgebra. *Indag. Math.* **25**, 225–231 (1963)
- [6] Gratzer, G., Jónsson, B., Lakser, H.: The amalgamation property in equational classes of modular lattices. *Pacific J. Math.* **45**, 507–524 (1973)
- [7] Horváth, G., Kátai-Urbán, K., Pach, P. P., Pluhár, G., Szabó, Cs., Pongrácz, A.: The number of monounary algebras. *Algebra Univ.* **66**, 81 (2011)

- [8] Isaacs, I.M., Zieschang, T.: Generating symmetric groups. *Am. Math. Mon.* **102**, 734–739 (1995)
- [9] Jakubíková-Studenovská, D., Pócs, J.: *Monounary Algebras*, 1st edn. UPJŠ, Košice (2009)
- [10] Jakubíková-Studenovská, D., Pócs, J.: Formations of finite monounary algebras. *Algebra Univ.* **68**, 249–255 (2012)
- [11] Jakubíková-Studenovská, D., Pöschel, R., Radeleczki, S.: Irreducible quasiorders of monounary algebras. *J. Aust. Math. Soc.* **93**, 259–276 (2012)
- [12] Jónsson, B.: Universal relational systems. *Math. Scand.* **4**, 193–208 (1957)
- [13] Jónsson, B.: The amalgamation property in varieties of modular lattices. *Not. Am. Math. Soc.* **18**, 400 (1971)
- [14] Káтай-Urbán, K., Pongrácz, A., Szabó, Cs.: The fine- and generative spectra of varieties of monounary algebras. *Algebra Univ.* **80**, 22 (2019)
- [15] Khaled M.: Algebras of concepts and their networks: Boolean algebras. An abstract in *Logic Colloquium 2021*, Adam Mickiewicz University, Poznań, Poland, 9–24 July (2021)
- [16] Khaled, M., Székely G.: Algebras of concepts and their networks. In: *Progress in Intelligent Decision Science. IDS 2020. Adv. Intell. Syst. Comput.*, vol. 1301. Springer, Cham (2021)
- [17] Khaled, M., Székely, G., Lefever, K., Friend, M.: Distances between formal theories. *Rev. Symb. Log.* **13**, 633–654 (2020)
- [18] Kiss, E. W., Márki, L., Pröhle, P., Tholen, W.: Categorical algebraic properties. A compendium on amalgamation, congruence extension, epimorphisms, residual smallness, and injectivity. *Stud. Sci. Math. Hung.* **18**, 79–141 (1983)
- [19] Lefever, K., Székely, G.: Comparing classical and relativistic kinematics in first-order logic. *Log. Anal.* **61**, 57–117 (2018)
- [20] Madárasz, J., Sayed-Ahmed, T.: Amalgamation, interpolation and epimorphisms in algebraic logic. *Algebra Univ.* **56**, 179–210 (2007)
- [21] McKenzie, R., McNulty, G.F., Taylor W.: *Algebras, Lattices, Varieties. Vol. I. The Wadsworth and Brooks/Cole Mathematics Series. Advanced Books and Software*, Monterey (1987)
- [22] Neumann, H.: Generalized free products with amalgamated subgroups. *Am. J. Math.* **70**, 590–625 (1948)
- [23] Némethi, I.: Cylindric-relativised set algebras have strong amalgamation. *J. Symb. Log.* **50**, 689–700 (1985)
- [24] Pigozzi, D.: Amalgamation, congruence-extension, and interpolation properties in algebras. *Algebra Univ.* **1**, 269–349 (1971)

- [25] Pierce, R.S.: Introduction to the Theory of Abstract Algebras. Holt, Rinehart and Winston, New York (1968)
- [26] Schreier, O.: Die Untergruppen der freien Gruppen. Abh. Math. Semin. Univ. Hambg. **5**, 161–183 (1927). (**German**)
- [27] Székely G.: Conceptual distance and algebras of concepts. An abstract in Logic Colloquium 2021, Adam Mickiewicz University, Poznań, Poland, 9–24 July (2021)
- [28] Vujošević, S.: Large subalgebras of a Boolean algebra. Publ. Inst. Math. **45**, 27–31 (1989)

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