Ricerche di Matematica (2025) 74:1467–1482
https://doi.org/10.1007/s11587-024-00912-2

Generalization of ρ-ideals associated with an m-system and a special radical class

Hatice Çay[1,3] · Alaa Abouhalaka[2] · Bayram Ali Ersoy[3]

Received: 31 July 2024 / Accepted: 14 November 2024 / Published online: 22 November 2024

Abstract

Let $\phi \neq S \subseteq R$ be an m-system of a ring R, and let ρ be a special radical. This study introduces the concept of S-ρ-ideals in noncommutative rings. This notion extends the previously studied ρ-ideals and can also be seen as a generalization of the right S-prime ideals. We show how some properties associated with ρ-ideals have evolved into results within these generalizations. Relationships between S-ρ-ideals and other types of ideals like ρ-ideals, right S-prime ideals, and S-finite ideals are shown. We show the behaviour of this notion in related rings. The construction of $(S \boxplus M)$-ρ-ideals in idealization rings is presented for an R-R-bimodule M. Additionally, we introduce S-$\mathcal{P}$-ideals using Baer-McCoy radical $\mathcal{P}$ and examine their properties.

Keywords prime ideal · S-prime ideal · m-system · Special radical classes · Noncommutative ring

Mathematics Subject Classification 16N20 · 16N40 · 16N60 · 16N80 · 16W99

1 Introduction

Let A be a noncommutative ring, and S be an m-system of A.

A. Abouhalaka, B. A. Ersoy have contributed equally to this work.

✉ Hatice Çay
hcay@medipol.edu.tr

Alaa Abouhalaka
alaa1aclids@gmail.com

Bayram Ali Ersoy
ersoya@gmail.com

1 IMU Vocational School, Istanbul Medipol University, 34810 Beykoz, Istanbul, Turkey

2 Department of Mathematics, Çukurova University, Balcalı, 01330 Adana, Turkey

3 Department of Mathematics, Yildiz Technical University, 34220 Esenler, Istanbul, Turkey

Amitsur and Kurosh were the first to independently find that all classical radicals have some shared characteristics, see [14]. Let ρ be a class of rings satisfying the following conditions:

(1) If $A \in \rho$ and $f : A \to B$ is a ring epimorphism then $B \in \rho$.
(2) For each ring A the sum $\rho(A) = \sum\{P \lhd A : P \in \rho\} \in \rho$.
(3) $\rho(A/\rho(A)) = 0$ for each ring A.

Then, ρ is called a radical class (in the sense of Kurosh and Amitsur). Each radical class is an idempotent, complete Hoehnke radical, that is:

(i) $f(\rho(A)) \subseteq \rho(f(A))$ for any ring homomorphism $f\colon A \to A'$.
(ii) If $J \lhd A$ and $\rho(J) = J$ then $J \subseteq \rho(A)$.
(iii) $\rho(A/\rho(A)) = 0$.
(iv) For each ring A, $\rho(\rho(A)) = \rho(A)$.

According to Andrunakievich, in [10], if $\mathcal{M}$ is a class of rings, then it is called a special class, if $\mathcal{M}$ is hereditary ($J \lhd A \in \rho \Rightarrow J \in \rho$) consists of prime rings and satisfies the condition: If $0 \neq J \lhd A$, $J \in \mathcal{M}$, and A is a prime ring then $A \in \mathcal{M}$ [14]. The class $\mathcal{U}(\mathcal{M}) = \{A\colon A$ has no nonzero homomorphic image in $\mathcal{M}\}$ constitutes a radical class of rings known as the upper radical class. A radical class ρ is termed a special radical, if it is the upper radical of a special class. Let $\rho = \mathcal{U}(\mathcal{M})$ be a special radical class, $\mathcal{S}_\rho = \{A\colon \rho(A) = 0\}$, and $\mathcal{K} = \{A\colon A$ is a prime ring$\}$. Then $\rho(A) = \cap\{J \lhd A : A/J \in \mathcal{K} \cap \mathcal{S}_\rho\}$.

Let ρ be a special radical. The concept of ρ-ideals (weakly ρ-ideals) in noncommutative ring is first introduced by Nico Groenewald [1] ([2]). In this context, a ρ-ideal P of a ring R is a proper ideal satisfying the condition that if $aRb \subseteq P$ and $a \notin \rho(R)$, then $b \in P$ for all $a, b \in R$. Recently, in [9], this concept has been generalized by defining right primary ρ-ideals and almost right primary ρ-ideals. The notion of ρ-ideal is also a generalization of J-ideal and n-ideal which were studied in commutative rings by Khashan and Bani-Ata [15], and Tekir et al. [3], respectively. In addition, in [4], ([6]), the concept of n-ideals (pure ideals) has been generalized to S-n-ideals (S-pure-), by taking S as a multiplicatively closed subset of a commutative ring. Actually, the study of mathematical concepts within noncommutative rings has captured considerable attention, especially in areas involving prime ideals, and radicals as seen in works such [16–23]. Many authors have focused on noncommutative S-versions, where S is a multiplicatively closed subset of R.

On the other hand, referring to [5], for the m-system S of R, an ideal P of R, disjoint from S, is called a right S-prime ideal if whenever $IJ \subseteq P$, implies either $I\langle s\rangle \subseteq P$ or $J\langle s\rangle \subseteq P$ for some ideals I, J of R and some (fixed) $s \in S$. It is worthy to note that defining S, in [5], as an *m-system* is not an option as we demonstrate in the following. Assume J is a prime ideal of a commutative ring R. Then, $S = R - J$ becomes a multiplicatively closed set disjoint from J. Using the definitions of S-prime and prime ideals in commutative rings, it follows that J is an $(R - J)$-prime ideal. However, for a prime ideal J of a noncommutative ring, the set $S = R - J$ becomes an m-system but not a multiplicatively closed set. Consequently, the prime ideal J cannot be considered $(R - J)$-prime, as the concept of $(R - J)$-prime is not applicable in noncommutative rings where $R - J$ is not necessarily a multiplicatively closed set.

In this work, we introduce the notion of S-ρ-ideal in noncommutative ring by taking S as an m-system. This new notion is a generalization of the ρ-ideals, that have been studied in [1], and also can be considered as a generalization of the concept of right S-prime ideals that have been studied in [5]. We show in Example 2 that our generalization is proper. We also discuss the relations between S-ρ-ideals and right S-prime ideals (Corollary 1, Remark 1 and Example 1). In addition, we show some connections between S-ρ-ideals and S-finite ideals (in the concept of [5]). In Theorem 12 we construct the $(S \boxplus M)$-ρ-ideal of the idealization ring $R \boxplus M$ (Nagata method of idealization), where M is a unitary R-R-bimodule M. In Section 3, we introduce the notion of S-$\mathcal{P}$-ideal by taking the special radical ρ as a prime radical $\mathcal{P}$ and obtain many properties of them. The ideal $I = M_n(\langle 12X \rangle)$ of the ring $R = M_n(\mathbb{Z}_{48}[X])$ is an S-$\mathcal{P}$-ideal which is not a $\mathcal{P}$-ideal, by taking the m-system $S = \{s, s^2, s^4, s^8, ...\}$, where $s = 6(e_{11} + e_{22} + ... + e_{nn})$, see Example 3. Moreover, the illustrative Example 5, shows that the ideal $P = M_2(\langle 4 \rangle)$ of the ring $R = M_2(\mathbb{Z}_{12})$ is not a $\mathcal{P}$-ideal, but it is an S_R-$\mathcal{P}$-ideal, where

$$S_R = \left\{ \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}; s \in S = \{1, 3, 9\} \right\}, \text{ and } s^* = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$

In Theorems 23, and 24 we study the stability of S-ρ-ideals under the direct product of rings.

In this paper, unless stated otherwise, by R we mean an associative, noncommutative ring without identity. Additionally, when referring to an ideal, we specifically mean a two-sided ideal. In addition, the notions $U(R)$, $\mathcal{P}(R)$, $Z(R)$, $\mathcal{J}(R)$, denote the set of unit elements, Baer-McCoy radical of R, the set of zero divisors of R, Jacobson radical of R, respectively. Throughout the paper, to avoid repetition and lengthiness, we will use the symbols S and ρ to represent an m-system and a special radical of a ring R, respectively, even if it is not explicitly stated each time.

2 S-ρ-ideals

2.1 Properties of S-ρ-ideals

In the following, we present the definition of S-ρ-ideal.

Definition 1 Let R be a ring. An ideal P of R, that is disjoint from S, is called S-ρ-ideal if and only if whenever $IJ \subseteq P$, then either $I\langle s \rangle \subseteq \rho(R)$ or $J\langle s \rangle \subseteq P$ for all ideals I, J of R, and for some $s \in S$.

If P is a ρ-ideal such that $P \cap S = \phi$, then it is also an S-ρ-ideal. The classes of ρ-ideals and S-ρ-ideals coincide when $S \subseteq U(R)$. The following proposition shows equivalent definitions of the concept of S-ρ-ideal.

Proposition 1 *Let P be an ideal of a ring R that has an identity. If $P \cap S = \phi$, then, the followings are equivalent:*

(1) *P is an S-ρ-ideal.*

(2) *For all* $x, y \in R$ *with* $\langle x\rangle\langle y\rangle \subseteq P$, *either* $\langle x\rangle\langle s\rangle \subseteq \rho(R)$ *or* $\langle y\rangle\langle s\rangle \subseteq P$ *for some* $s \in S$.
(3) *For all* $x, y \in R$ *with* $xRy \subseteq P$, *either* $x\langle s\rangle \subseteq \rho(R)$ *or* $y\langle s\rangle \subseteq P$ *for some* $s \in S$.

Proof (1) $\Rightarrow$ (2) Clear by the Definition 1.
(2) $\Rightarrow$ (3) Let $xRy \subseteq P$ for some $x, y \in R$. Then, $\langle x\rangle\langle y\rangle \subseteq P$, hence, by (2), either $x\langle s\rangle \subseteq \langle x\rangle\langle s\rangle \subseteq \rho(R)$, or $y\langle s\rangle \subseteq \langle y\rangle\langle s\rangle \subseteq P$ for some $s \in S$.
(3) $\Rightarrow$ (1) Suppose $IJ \subseteq P$, for some ideals I, J of R. If $I\langle s\rangle \nsubseteq \rho(R)$, then, there exists $a \in I$ such that $a\langle s\rangle \nsubseteq \rho(R)$. For all $b \in J$, we have $aRb \subseteq IJ \subseteq P$, hence, by (3), we have $b\langle s\rangle \subseteq P$, and thus, $J\langle s\rangle \subseteq P$. That completes the proof. □

Corollary 1 *Let* $P \lhd R$ *such that* $P \cap S = \phi$. *If* P *is a right* S*-prime ideal such that* $P \subseteq \rho(R)$, *then,* P *is an* S-ρ*-ideal.*

Proof Let $xRy \subseteq P$ and assume $x\langle s\rangle \nsubseteq \rho(R)$ for $x, y \in R$. Then, $x\langle s\rangle \nsubseteq P$. Since P is a right S-prime ideal, then $y\langle s\rangle \subseteq P$. Thus by (3) of Proposition 1, P is an S-ρ-ideal. □

Remark 1 The condition $P \subseteq \rho(R)$ in Corollary 1 can not be omitted in general. It is shown by the following example.

Example 1 Let $R = M_n(\mathbb{Z}_{48}[X])$. Then since R is finite, it is an Artinian ring hence the Jacobson and the Baer-McCoy radicals of R are equal, and by Theorem 10.21 in [8] we know that $\mathcal{P}(M_n(R)) = M_n(\mathcal{P}(R))$, also from Theorem 3 of [7], $\mathcal{P}(R[X]) = \mathcal{P}(R)[X]$. Hence,

$$\mathcal{P}(R) = \mathcal{P}(M_n(\mathbb{Z}_{48}[X])) = M_n(\mathcal{P}((\mathbb{Z}_{48}[X])) = M_n(\mathcal{P}(\mathbb{Z}_{48})[X])) = M_n(\langle 6\rangle[X])$$

Consider the m-system: $S = \{s, s^2, s^4, s^8, ...\}$, where $s = 8(e_{11} + e_{22} + ... + e_{nn})$, and $I = M_n(\langle 8X\rangle)$. It is clear that $S \cap I = \phi$ and $I \nsubseteq \mathcal{P}(R)$. Now let $\mathcal{N}_1 R\mathcal{N}_2 \subseteq I$, for some matrices $\mathcal{N}_1, \mathcal{N}_2$ in R. Then, there exist some matrices $\mathcal{I}_1, \mathcal{I}_2$, such that either $\mathcal{N}_1 = x\mathcal{I}_1$ or $\mathcal{N}_2 = x\mathcal{I}_2$. Hence, either $\mathcal{N}_1\langle s\rangle \subseteq I$ or $\mathcal{N}_2\langle s\rangle \subseteq I$. Thus I is a right S-prime ideal by Proposition 2.7 of [1]. However, by taking the ideals $A_1 = M_n(\langle 4X\rangle), A_2 = M_n(\langle 2\rangle)$, we have $A_1A_2 \subseteq I$, however, $A_1\langle s'\rangle \nsubseteq \mathcal{P}(R)$ and $A_2\langle s'\rangle \nsubseteq I$, for each $s' \in S$. Hence, I is not an S-$\mathcal{P}$-ideal. Thus, the condition $I \subseteq \rho(R)$, in Corollary 1, can not be omitted.

The subsequent examples show that this new notion (S-ρ-ideal) is a proper generalization of the concept of ρ-ideal.

Example 2 Let R be a prime ring, and $\rho = \mathcal{P}$. Notice that the ring $M_n(R)$ is also prime. Take $I = M_n(\langle 0\rangle)$ and let

$$S_{M_n(R)} = \{s'; s' = s\,(e_{11} + \cdots + e_{nn})\,; s \in S\}.$$

It is easy to check that $S_{M_n(R)}$ is an m-system of the ring $M_n(R)$, and $S_{M_n(R)} \cap I = \phi$. Suppose $J, K \lhd M_n(R)$ such that $JK \subseteq I$. Assume $J\langle s'\rangle \nsubseteq \rho(R)$ for all $s' \in S_{M_n(R)}$. Then, since I is prime, either $J \subseteq I \subseteq \rho(R)$, contradiction, or $K \subseteq I$. Thus, $K\langle s'\rangle \subseteq I$ for some $s' \in S_{M_n(R)}$, and hence, I is an S-ρ-ideal.

Remark 2 The zero ideal is an S-ρ-ideal of the prime ring.

Example 3 Let $R = M_n(\mathbb{Z}_{48}[X])$. Then by the explanation in Example 1, we have $\mathcal{P}(R) = M_n(\langle 6\rangle[X])$. Take the m-system $S = \{s, s^2, s^4, s^8, ...\}$, where $s = 6(e_{11} + e_{22} + ... + e_{nn})$. Consider the ideals $A_1 = M_n(\langle 4\rangle)$, $A_2 = M_n(\langle 3X\rangle)$, and $I = M_n(\langle 12X\rangle)$, then, $A_1A_2 \subseteq I$ but $A_1 \nsubseteq \rho(R)$, $A_2 \nsubseteq I$. Thus, I is not a $\mathcal{P}$-ideal. If $\mathcal{N}_1, \mathcal{N}_2 \in R$ with $\mathcal{N}_1 R \mathcal{N}_2 \subseteq I$, then either $\mathcal{N}_1 = xI_1$ or $\mathcal{N}_2 = xI_2$ for some matrices I_1, I_2 of R. Then, either $\mathcal{N}_1\langle s^2\rangle \subseteq I$ or $\mathcal{N}_2\langle s^2\rangle \subseteq I$. Hence I is a right S-prime ideal. Since $I \subseteq \mathcal{P}(R)$, then, by Corollary 1, I is an S-$\mathcal{P}$-ideal.

In referring to [10], for an ideal J of any ring R, $\rho(R) \subseteq \rho^*(J)$ and $J \subseteq \rho^*(J)$, where $\rho^*(J)$ is the radical of the ideal J. Moreover, $\rho(R/J) = \rho^*(J)/J$. In fact, $J \subseteq \rho(R)$ if and only if $\rho(R) = \rho^*(J)$; i.e. if and only if $\rho(R/J) = \rho(R)/J$.

Proposition 2 *Let P be an ideal of a ring R with identity, such that $(\rho(R) : \langle s\rangle) = \rho(R)$ for some $s \in S$. Then, the followings are equivalent.*

(1) *P is an S-ρ-ideal of R associated with $s \in S$.*
(2) *(a) $P \subseteq (\rho(R) : \langle s\rangle)$.*
(b) For all $x, y \in R$ with $xRy \subseteq P$, either $x \in (\rho^*(P) : \langle s\rangle)$ or $y \in (P : \langle s\rangle)$.

Proof (1) $\Rightarrow$ (2) (a) Suppose $P \nsubseteq (\rho(R) : \langle s\rangle)$, then, for all $x \in P\backslash(\rho(R) : \langle s\rangle)$ and any $y \in R$ we have $\langle x\rangle\langle y\rangle \subseteq P$, hence, by (1), we obtain $y \in (P : \langle s\rangle)$ for some $s \in S$, and thus, $R = (P : \langle s\rangle)$, which implies $1\langle s\rangle \subseteq P$, a contradiction since $P \cap S = \phi$. Thus, $P \subseteq (\rho(R) : \langle s\rangle)$.

(b) Now assume $xRy \subseteq P$ for some x, y of R. By (3) of Proposition 1, either $x \in (\rho(R) : \langle s\rangle) \subseteq (\rho^*(P) : \langle s\rangle)$ or $y \in (P : \langle s\rangle)$ for some $s \in S$.

(2) $\Rightarrow$ (1) Assume $A_1A_2 \subseteq P$ for some ideals A_1, A_2 of R. Assume that $A_1\langle s\rangle \nsubseteq \rho(R)$ and $A_2\langle s\rangle \nsubseteq P$ for all $s \in S$. Then, there is $a_1 \in A_1\backslash(\rho(R) : \langle s\rangle)$ and $a_2 \in A_2\backslash(P : \langle s\rangle)$. Hence, $a_1Ra_2 \subseteq A_1A_2 \subseteq P$, and by (b) we obtain either $a_1 \in (\rho^*(P) : \langle s\rangle)$ or $a_2 \in (P : \langle s\rangle)$. Now since $P \subseteq (\rho(R) : \langle s\rangle) = \rho(R)$, then, $\rho^*(P) = \rho(R)$. Thus, either $a - 1 \in (\rho(R) : \langle s\rangle)$ or $a_2 \in (P : \langle s\rangle)$, a contradiction in both cases. Hence, P is an S-ρ-ideal of R. $\square$

Corollary 2 *If $\{J_i : i \in \Delta\}$ is a nonempty set of S-ρ-ideals of a ring R. Then $\bigcap J_i$ is S-ρ-ideal.*

Proof Let $xRy \subseteq \bigcap_{i\in\Delta} J_i$ with $x\langle s\rangle \nsubseteq \rho(R)$ for $x, y \in R$. Then $xRy \subseteq J_i$ for each $i \in \Delta$. Because J_i is an S-ρ-ideal, we get $y\langle s\rangle \subseteq J_i$ for each $i \in \Delta$. Thus $y\langle s\rangle \subseteq \bigcap_{i\in\Delta} J_i$ that completes the proof. $\square$

Proposition 3 *Let R be a ring with identity and $P \lhd R$ such that $S \cap P = \phi$.*

(1) *If P is an S-ρ-ideal, then, $P \subseteq (\rho(R) : \langle s\rangle)$ for some $s \in S$.*
(2) *$\rho(R)$ is an S-ρ-ideal if and only if $\rho(R)$ is a right S-prime ideal.*

Proof (1) Assume P is an S-ρ-ideal. For all $a \in P$, $\langle a\rangle\langle s_1\rangle \subseteq P$, for some $s_1 \in S$. Hence, either $\langle a\rangle\langle s\rangle \subseteq \rho(R)$ or $\langle s_1\rangle\langle s\rangle \subseteq P$ for some $s \in S$. If $\langle s_1\rangle\langle s\rangle \subseteq P$, then, for some $r \in R$, $s_2 = s_1rs \in S$, hence, $\langle s_2\rangle \subseteq \langle s_1\rangle\langle s\rangle \subseteq P$, contradiction because $S \cap P = \phi$. Hence, $\langle a\rangle\langle s\rangle \subseteq \rho(R)$ for all $a \in P$, and thus, $P \subseteq (\rho(R) : \langle s\rangle)$.

(2) The proof is routine. $\square$

Corollary 3 *[Proposition 1.5 of [1]] If P is a ρ-ideal, then, $P \subseteq \rho(R)$.*

Proof The proof follows by (1) of Proposition 3, by taking $S = \{1\}$. Recall that $(P : R) = P$. $\square$

Corollary 4 *[(1) of Corollary 1.14 of [1]] For a ring R we have that $\rho(R)$ is a ρ-ideal if and only if $\rho(R)$ is a prime ideal.*

Proof The proof follows by (2) of Proposition 3, by taking $S = \{1\}$. $\square$

Corollary 5 *Let ρ_1 and ρ_2 be two special radicals such that $\rho_1 \leq \rho_2$, then each S-ρ_1-ideal is an S-ρ_2-ideal.*

Proof Let P be an S-ρ_1-ideal and $aRb \subseteq P$ and $a\langle s\rangle \nsubseteq \rho_2(R)$. Since $\rho_1 \leq \rho_2$ we get $\rho_1(R) \subseteq \rho_2(R)$ so $a\langle s\rangle \nsubseteq \rho_1(R)$. Since P is an S-ρ_1-ideal then by Proposition 1 (3) we obtain $b\langle s\rangle \subseteq P$. Hence, P is an S-ρ_2-ideal. $\square$

Theorem 4 *Let R be a ring with identity and $P \lhd R$ such that $P \cap S = \phi$. Then, $(P : \langle s\rangle)$ is an S-ρ-ideal, if and only if, P is an S-ρ-ideal, for some $s \in S$.*

Proof Suppose $(P : \langle s\rangle)$ is an S-ρ-ideal, and let $xRy \subseteq P$ for some $x, y \in R$, then, $xRy \subseteq (P : \langle s\rangle)$, hence by assumption, either $x\langle s_1\rangle \subseteq \rho(R)$ or $y\langle s_1\rangle \subseteq (P : \langle s\rangle)$ for some $s_1 \in S$. If $y\langle s_1\rangle \subseteq (P : \langle s\rangle)$, then, $y\langle s_1\rangle\langle s\rangle \subseteq P$, hence there exists $s' = s_1rs$ for some $r \in R$ such that $y\langle s'\rangle \subseteq y\langle s_1\rangle\langle s\rangle \subseteq P$. If $x\langle s_1\rangle \subseteq \rho(R)$, then similarly, we find $x\langle s'\rangle \subseteq \rho(R)$. Thus, P is an S-ρ-ideal. Conversely, suppose P is an S-ρ-ideal, and let $xRy \subseteq (P : \langle s\rangle)$ for some $x, y \in R$, then, $\langle x\rangle\langle y\rangle\langle s\rangle \subseteq P$, hence, by assumption, either $\langle x\rangle\langle s_1\rangle \subseteq \rho(R)$ or $\langle y\rangle\langle s\rangle\langle s_1\rangle \subseteq P$ for some $s_1 \in S$. If $\langle y\rangle\langle s\rangle\langle s_1\rangle \subseteq P$, then, $\langle y\rangle\langle s\rangle\langle s_1\rangle \subseteq (P : \langle s\rangle)$, hence there exists $s' = srs_1$ for some $r \in R$ such that $y\langle s'\rangle \subseteq \langle y\rangle\langle s\rangle\langle s_1\rangle \subseteq (P : \langle s\rangle)$. If $\langle x\rangle\langle s_1\rangle \subseteq \rho(R)$, then similarly, we find $x\langle s'\rangle \subseteq \rho(R)$. Thus, $(P : \langle s\rangle)$ is an S-ρ-ideal. $\square$

Lemma 2.1 of [5] states that if S is an m-system of a ring R, then for some ring epimorphism $f\colon R \to f(R)$, $f(S)$ is an m-system of $f(R)$. In addition, the set $\bar{S} = \{\bar{s} = s + A : s \in S\}$, is an m-system of R/A for any ideal A of R. In the following, we show how the property of being S-ρ-ideal is maintained when transferring to related rings.

Theorem 5 *Let $f\colon R_1 \to R_2$ be a ring epimorphism, and P be an S-ρ-ideal of R_1 contains the kernel of f. Then, $f(P)$ is an $f(S)$-ρ-ideal of R_2.*

Proof Let $xR_2y \subseteq f(P)$ and $x\langle f(s)\rangle \nsubseteq \rho(R_2)$ for $x, y \in R_2$. Since f is an epimorphism, there exist $c, d \in R_1$ such that $f(c) = x$ and $f(d) = y$. Thus, $xR_2y = f(c)f(R_1)f(d) = f(cR_1d) \subseteq f(P)$. Since $\mathrm{Ker}(f) \subseteq P$ we obtain $cR_1d \subseteq P$. Since P is an S-ρ-ideal of R_1, then, either $c\langle s\rangle \subseteq \rho(R_1)$ or $d\langle s\rangle \subseteq P$ for some $s \in S$. If $c\langle s\rangle \subseteq \rho(R_1)$, then, since ρ is a complete, idempotent Hoehnke radical, we have $f(\rho(R_1)) \subseteq \rho(R_2)$. Hence, $x\langle f(s)\rangle = f(c)\langle f(s)\rangle \subseteq f(\rho(R_1)) \subseteq \rho(R_2)$ that is a contradiction. Hence, $c\langle s\rangle \nsubseteq \rho(R_1)$ and thus, $d\langle s\rangle \subseteq P$. Since $y = f(d)$, then $y\langle f(s)\rangle \subseteq f(P)$, consequently, $f(P)$ is an $f(S)$-ρ-ideal of R_2. $\square$

Corollary 6 *For a ring epimorphism $f : R_1 \to R_2$, and an ideal J of R_2. If $f^{-1}(J)$ is an S-ρ-ideal of R_1, then, J is an $f(S)$-ρ-ideal of R_2.*

Proof Because the preimage of any ideal of R_2 under f is an ideal of R_1 contains the kernel of f, the proof follows from Theorem 5. $\square$

Theorem 6 *Let S be an m-system of R_1, $f\colon R_1 \to R_2$ be a rings epimorphism, and $P \lhd R$ such that $Ker(f) \subseteq P \cap \rho(R)$ and $P \cap S = \phi$. If $f(P)$ is an $f(S)$-ρ ideal of R_2, then P is an S-ρ-ideal of R_1.*

Proof Suppose that $A_1 B_1 \subseteq P$ for ideals A_1, B_1 of R. Then, $f(A_1)f(B_1) = f(A_1 B_1) \subseteq f(P)$. Since $f(P)$ is an $f(S)$-ρ ideal of R_2, then either

$$f(B_1)\langle f(s)\rangle \subseteq f(P),$$

for some $s \in S$, or $f(A_1)\langle f(s)\rangle \subseteq \rho(R_2)$.

If $f(B_1)\langle f(s)\rangle \subseteq f(P)$, then $B_1\langle s\rangle \subseteq f^{-1}(f(B_1\langle s\rangle) \subseteq f^{-1}(f(P)) = P$, since $\mathrm{Ker}(f) \subseteq P$. If $f(A_1)\langle f(s)\rangle \subseteq \rho(R_2)$, in this case we show that $A_1\langle s\rangle \subseteq \rho(R_1)$ as the following. Let J be an ideal of R_1 with $R_1/J \in \mathcal{S}_\rho \cap \mathcal{K}$, then $\rho(R_1) \subseteq J$, thus $\mathrm{Ker}(f) \subseteq J$. Hence, $f(R_1)/f(J) \cong (R_1/\mathrm{Ker}(f))/(J/\mathrm{Ker}(f)) \cong R_1/J$, so $R_2/f(J) = f(R_1)/f(J) \in \mathcal{S}_\rho \cap \mathcal{K}$, and thus $f(A_1)\langle f(s)\rangle \subseteq \rho(R_2) \subseteq f(J)$, thus $A_1\langle s\rangle \subseteq f^{-1}(f(A_1\langle s\rangle)) \subseteq f^{-1}(f(J)) = J$, for each J in $\cap\{I \lhd R : R/I \in \mathcal{S}_\rho \cap \mathcal{K}\}$, and hence $A_1\langle s\rangle \subseteq \rho(R_1)$. Thus, P is an S-ρ-ideal of R_1. $\square$

Corollary 7 *Let $f : R_1 \to R_2$ be a ring epimorphism, and A be an $f(S)$-ρ ideal of R_2. If $Ker(f) \subseteq \rho(R)$, then $f^{-1}(A)$ is an S-ρ-ideal of R_1.*

Proof Let $J = f^{-1}(A)$. Then, J is an S-ρ-ideal of R_1. by Theorem 6, since $\mathrm{Ker}(f) \subseteq J \cap \rho(R)$ and $f(J) = f(f^{-1}(A)) = A$ is a an $f(S)$-ρ ideal of R_2. $\square$

Theorem 7 *Let $A, B \lhd R$ such that $A \subseteq B$. Then, we get the followings:*

(1) *If B is an S-ρ-ideal of R, then, B/A is an $\overline{S}$-ρ-ideal of R/A.*
(2) *If B/A is an $\overline{S}$-ρ-ideal of R/A, and $A \subseteq \rho(R)$, then, B is an S-ρ-ideal of R.*
(3) *If B/A is an $\overline{S}$-ρ-ideal of R/A, and A is a ρ-ideal of R, then, B is an S-ρ-ideal of R.*
(4) *If R_1 is a subring of R_2 and I_2 is an S-ρ-ideal of R_2, then $I_2 \cap R_1$ is an S-ρ-ideal of R_1.*

Proof (1) Suppose B is an S-ρ-ideal of R. Since $A \cap S = \phi$, then it is easy to show that $(B/A) \cap \overline{S} = \phi$. Take the canonical epimorphism $f\colon R \to R/A$, since, $\mathrm{Ker}(f) = A \subseteq B$, then, by Theorem 5 we obtain that $f(B) = B/A$ is an $\overline{S}$-ρ-ideal of R/A.

(2) Assume $A \subseteq \rho(R)$ and B/A is an $\overline{S}$-ρ-ideal of R/A. Notice that $A \cap S = \phi$, because $(B/A) \cap \overline{S} = \phi$. By Theorem 6, B is an S-ρ-ideal.

(3) Since A is a ρ-ideal, then, by Proposition 1.5 of [1], $A \subseteq \rho(R)$. Hence, by item (2), B is an S-ρ-ideal of R.

(4) Consider the canonical injection. $i : R_1 \to R_2$, then, $\mathrm{Ker}(i) = 0 \subseteq \rho(R)$, hence, by Corollary 7, $i^{-1}(I_2) = I_2 \cap R_1$ is an S-ρ-ideal of R_1.

$\square$

Let S_1 and S_2 be two m-systems of R with $S_1 \subseteq S_2$, then, clearly, every S_1-ρ-ideal of R disjoint from S_2, is an S_2-ρ-ideal. However, the converse is not true in general. The converse holds under special conditions and we show that as follows.

Proposition 8 *Let S_1, S_2 be two m-systems of R with $S_1 \subseteq S_2$, and for every $a \in S_2$ there exists $b \in S_2$ such that $arb \in S_1$ for some $r \in R$. If P is an S_2-ρ-ideal of R, then P is an S_1-ρ-ideal.*

Proof Assume $xRy \subseteq P$. Then $x\langle s_1\rangle \subseteq \rho(R)$ or $y\langle s_1\rangle \subseteq P$ for some $s_1 \in S_2$. By assumption, there exists some $s_2 \in S_2$ such that $s = s_1 r s_2 \in S_1$ for some $r \in R$. Hence, $x\langle s\rangle \subseteq x\langle s_1\rangle\langle s_2\rangle \subseteq \rho(R)$ or $y\langle s\rangle \subseteq y\langle s_1\rangle\langle s_2\rangle \subseteq P$. □

Proposition 9 *Let P be an S-ρ-ideal of R. If I is an ideal of R meets S, then the following holds:*

(1) *$I \cap P$ is an S-ρ-ideal of R.*
(2) *PI is an S-ρ-ideal of R.*

Proof (1) Let $A_1A_2 \subseteq (P \cap I)$ for some ideals A_1, A_2 of R. Then, $A_1A_2 \subseteq P$, hence, either $A_1\langle s\rangle \subseteq \rho(R)$ or $A_2\langle s\rangle \subseteq P$ for some $s \in S$. If $A_2\langle s\rangle \subseteq P$, then for some $x \in (I \cap S)$, $A_2RsRxR \subseteq (P \cap I)$. Since s, x in the m-system S, there exists $r \in R$ such that $s_1 = srx$, and hence $A_2\langle s_1\rangle = A_2Rs_1R \subseteq A_2RsRxR \subseteq (P \cap I)$. If $A_1\langle s\rangle \subseteq \rho(R)$, then similarly, we obtain $A_1\langle s_1\rangle \subseteq \rho(R)$. Hence, $(P \cap I)$ is an S-ρ-ideal associated with $s_1 \in S$, since $(P \cap I) \cap S = \phi$.

(2) Similar to (1). □

Theorem 10 *Let $P \lhd R$ with $S \cap P = \phi$, and $P \subseteq \rho(R)$.*

(1) *P is an S-ρ-ideal.*
(2) *For $P_1, P_2, ..., P_n \lhd R$, with $P_1P_2 \cdots P_n \subseteq P$, $P_i\langle s\rangle \subseteq \rho(R)$ for some $s \in S$, and $i \in \{1, ..., n\}$.*
(3) *If for elements $x_1, x_2, ..., x_n \in R$, $x_1Rx_2R...Rx_n \subseteq P$, then $x_k\langle s\rangle \subseteq \rho(R)$ for $k \in \{1, ..., n\}$. Then,* (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Proof (1) $\Rightarrow$ (2) Assume $P_1P_2 \cdots P_n \subseteq P$, for $P_1, P_2, ..., P_n \lhd R$. Since P is an S-ρ-ideal then either $P_1\langle s\rangle \subseteq \rho(R)$ or $P_2, ...P_n\langle s\rangle \subseteq P$. If $P_1\langle s\rangle \subseteq \rho(R)$, then, nothing to prove. If $P_2 \cdots P_n\langle s\rangle \subseteq P$, then either $P_2\langle s\rangle \subseteq \rho(R)$ or $P_3 \cdots P_n\langle s\rangle\langle s\rangle \subseteq P$. If $P_2\langle s\rangle \subseteq \rho(R)$, then, nothing to prove. If $P_3 \cdots P_n\langle s\rangle\langle s\rangle \subseteq P$, then, by repeating the process above, we obtain either $P_n\langle s\rangle \subseteq \rho(R)$ or $\langle s\rangle^n \subseteq P$. If $\langle s\rangle^n \subseteq P$, then, $s_1 = sr_1sr_2s \cdots r_{n-1}s \in S$, for some $r_k \in R$ where $k \in \{1, 2, \cdots, n-1\}$, hence, $\langle s_1\rangle \subseteq \langle s\rangle^n \subseteq P$, a contradiction since $P \cap S = \phi$.

(2) $\Rightarrow$ (3) If we take $x_1Rx_2R...Rx_n \subseteq \langle x_1\rangle...\langle x_n\rangle \subseteq P$, then taking $P_i = \langle x_i\rangle$ in the condition (2) we get the result. □

2.2 *S*-finite and *S*-ρ-ideals

Recall that Definition 3.1 (1) of [5], states that an ideal P of a ring R is called S-finite, if there exists a finitely generated ideal I such that $P\langle s\rangle \subseteq I \subseteq P$ for some $s \in S$. In the following, we show some connections between S-ρ and S-finite ideals.

Corollary 8 *Let J be an ideal of R with $J \nsubseteq (\rho(R) : \langle s\rangle)$ for all $s \in S$. If P is an ideal of R such that JP is a finitely generated S-ρ-ideal of R, then, P is S-finite.*

Proof Since $JP \subseteq JP$, $J \nsubseteq (\rho(R) : \langle s\rangle)$, and JP is an S-ρ-ideal, then, $P\langle s_1\rangle \subseteq JP \subseteq P$ for some $s_1 \in S$. Thus, P is S-finite. □

Corollary 9 *Let J be an ideal of R with $J \nsubseteq (\rho(R) : \langle s\rangle)$ for all $s \in S$, and let P_1, P_2 be ideals of R such that $JP_1 = JP_2$. If $P_1 \cap P_2$ is finitely generated, and P_i is an S-ρ-ideal of R for $i \in \{1, 2\}$, then, P_j is S-finite for $i \neq j \in \{1, 2\}$.*

Proof Without loss of generality, suppose that P_2 is an S-ρ-ideal. Since $JP_1 \subseteq P_2$, $J \nsubseteq (\rho(R) : \langle s\rangle)$, and P_2 is an S-ρ-ideal, then, $P_1\langle s_1\rangle \subseteq P_2$ for some $s_1 \in S$. Hence, $P_1\langle s_1\rangle \subseteq P_1 \cap P_2 \subseteq P_1$. Thus, P_1 is S-finite. Similarly, we show P_2 is S-finite, if P_1 is an S-ρ-ideal. □

In referring to [13], the pseudo radical of an ideal J of a ring R ($\sqrt{J}$), equals to the sum of all ideals U of R such that $U^k \subseteq J$ for some $k \in \mathbb{Z}^+$. Notice that $\sqrt{J} \subseteq \mathcal{P}^*(J)$.

Corollary 10 *Let P_1, P_2 be ideals of R such that $P_1 \nsubseteq (\rho(R) : \langle s\rangle)$ for all $s \in S$. If $\sqrt{P_1P_2}$ is a finitely generated S-ρ-ideal, then, $\sqrt{P_2}$ is S-finite.*

Proof Suppose that $\sqrt{P_1P_2}$ is an S-ρ-ideal. Then, by Lemma 1.2 of [13], $\sqrt{P_1}\sqrt{P_2} \subseteq \sqrt{P_1} \cap \sqrt{P_2} = \sqrt{P_1 \cap P_2} = \sqrt{P_1P_2}$. Thus, by assumption, either $\sqrt{P_1}\langle s\rangle \subseteq \rho(R)$, which implies $P_1 \subseteq (\rho(R) : \langle s\rangle)$, a contradiction, or $\sqrt{P_2}\langle s\rangle \subseteq \sqrt{P_1P_2} = \sqrt{P_1} \cap \sqrt{P_2} \subseteq \sqrt{P_2}$, hence $\sqrt{P_2}$ is S-finite. □

Proposition 11 *Let $P \lhd R$ such that $R/P \in \mathcal{S}_\rho \cap \mathcal{K}$. Then, the followings hold:*

(1) *If P be an S-ρ-ideal, and $\rho(R)$ is finitely generated, then P is S-finite.*
(2) *If $P = (\rho(R) : \langle s\rangle)$, then, P is an S-ρ-ideal.*

Proof (1) Since $R/P \in \mathcal{S}_\rho \cap \mathcal{K}$, and $\rho(R) = \cap\{J \lhd R : R/J \in \mathcal{K} \cap \mathcal{S}_\rho\}$ then, $\rho(R) \subseteq P$. In addition, by (1) of Proposition 3, we get $P \subseteq (\rho(R) : \langle s\rangle)$ for some $s \in S$. Thus, $P\langle s\rangle \subseteq \rho(R) \subseteq P$, consequently, P is S-finite.

(2) Let $aRb \subseteq P$ for some $a, b \in R$. If $a\langle s\rangle \nsubseteq \rho(R)$ for all $s \in S$, then, $a\langle s\rangle \nsubseteq P$, however P is prime, and hence, $b \in P$ consequently, $b\langle s\rangle \subseteq P$, and P is an S-ρ-ideal. □

2.3 Idealization

Let R be a ring, for a unitary R-R-bimodule M, the idealization of M in R is the ring $R \boxplus M = \{(r, m) : r \in R, m \in M\}$, with the usual addition and the multiplication defined as:

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1),$$

for all $(r_1, m_1), (r_2, m_2) \in R \boxplus M$. Therefore, one can easily check that if S is an m-system of R, then $S_M = S \boxplus M$ is an m-system of $R \boxplus M$. Moreover, it is well-known fact that if P is an ideal of R and N is an R-R-bisubmodule of M, then, $P \boxplus N$ is an ideal of $R \boxplus M$ if and only if $PM + MP \subseteq N$. In addition, in referring to [11], if ρ is a special radical, then, $\rho(R \boxplus M) = \rho(R) \boxplus M$.

Theorem 12 *Let M be an R-R-bimodule, and $J \lhd R$. Then, $J \boxplus M$ is an $(S \boxplus M)$-ρ-ideal of $R \boxplus M$ if and only if J is an S-ρ-ideal of R.*

Proof First notice that $(J \boxplus M) \cap (S \boxplus M) = \phi$, if and only if, $S \cap J = \phi$. Now assume $J \boxplus M$ is an $(S \boxplus M)$-ρ-ideal of $R \boxplus M$, and let $aRb \subseteq J$ for some $a, b \in R$, then, $(a, 0)R \boxplus M(b, 0) \subseteq J \boxplus M$, thus, by assumption,

$$\text{either } (a, 0)\langle(s, m)\rangle \subseteq \rho(R) \boxplus M \text{ or } (b, 0)\langle(s, m)\rangle \subseteq J \boxplus M,$$

for some $(s, m) \in S \boxplus M$. Hence, either $a\langle s\rangle \subseteq \rho(R)$ or $b\langle s\rangle \subseteq J$, consequently, J is an S-ρ-ideal of R.

Conversely, suppose J is an S-ρ-ideal of R, and let $(a, m)R \boxplus M(b, n) \subseteq J \boxplus M$ for some $(a, m), (b, n) \in (R) \boxplus M$. Then, $aRb \subseteq J$, hence, by assumption, either $a\langle s\rangle \subseteq \rho(R)$ or $b\langle s\rangle \subseteq J$ for some $s \in S$, and thus,

$$\text{either } (a, m)\langle(s, 0)\rangle \subseteq \rho(R) \boxplus M = \rho(R \boxplus M) \text{ or } (b, n)\langle(s, 0)\rangle \subseteq J \boxplus M.$$

Hence, $J \boxplus M$ is an $(S \boxplus M)$-ρ-ideal of $R \boxplus M$. □

3 S-$\mathcal{P}$-ideals

In this section, the special radical will be the prime radical, i.e., the (Baer's) lower nilradical or the Baer-McCoy radical of R, and we will denote it by $\mathcal{P}$. It is a well-known fact that the prime radical of R is the intersection of all prime ideals of R. While the Jacobson radical is denoted by $\mathcal{J}$.

Let R be a commutative ring with identity, and S be a multiplicatively closed subset of R. In referring to [4], Definition 1, states that an ideal I of R disjoint from S is called an S-n-ideal of R if there exists an (a fixed) $s \in S$ such that for all $x, y \in R$ with $xy \in I$ either $sx \in \sqrt{0}$ or $sy \in I$. Here $\sqrt{0}$ is the set of nilpotent elements of the ring R, which is also the intersection of its prime ideals, i.e., the prime radical of the ring R. In the following, we present our definition of the noncommutative case.

Definition 2 Let $\mathcal{P}$ be the prime radical of R. An ideal P of R, that is disjoint from S, is said to be S-$\mathcal{P}$-ideal if and only if whenever $IJ \subseteq P$, then either $I\langle s\rangle \subseteq \mathcal{P}(R)$ or $J\langle s\rangle \subseteq P$ for all ideals I, J of R, and for some $s \in S$.

Let S be an m-system of a ring R. It is easy to check that every $\mathcal{P}$-ideal, disjoint from S is an S-$\mathcal{P}$-ideal, and the classes of $\mathcal{P}$-ideals and S-$\mathcal{P}$-ideals coincide when $S \subseteq U(R)$.

Proposition 13 *Let R be a ring with an identity and $P \lhd R$ with $P \cap S = \phi$. The followings are equivalent:*

(1) *P is an S-$\mathcal{P}$-ideal.*
(2) *For all $x, y \in R$ with $\langle x\rangle\langle y\rangle \subseteq P$, either $\langle x\rangle\langle s\rangle \subseteq \mathcal{P}(R)$ or $\langle y\rangle\langle s\rangle \subseteq P$ for some $s \in S$.*

(3) *For all* $x, y \in R$ *with* $xRy \subseteq P$, *either* $x\langle s\rangle \subseteq \mathcal{P}(R)$ *or* $y\langle s\rangle \subseteq P$ *for some* $s \in S$.

Proof By taking ρ to be the prime radical, the proof follows from Proposition 1. □

Remark 3 If the ring R is commutative and unitary, then, the definition of S-n-ideal (Definition 1 of [4]) is equivalent to our definition of S-$\mathcal{P}$-ideal (item (3) of Proposition 13), when considering S as a multiplicatively closed subset of R, as we show in the following:

Suppose $I \lhd R$, which is disjoint from the multiplicatively closed subset S. If for all $x, y \in R$ with $xy \in I$ either $sx \in \sqrt{0}$ or $sy \in I$ for some $s \in S$, then, suppose that $xRy \subseteq I$, hence, $xy \in I$, which implies either $sx \in \sqrt{0}$ or $sy \in I$, thus, $x\langle s\rangle \subseteq \sqrt{0} = \mathcal{P}(R)$ or $y\langle s\rangle \subseteq I$, by (3) of Proposition 13.

Conversely, if for all $x, y \in R$ with $xRy \subseteq I$, either $x\langle s\rangle \subseteq \mathcal{P}(R)$ or $y\langle s\rangle \subseteq I$ for some $s \in S$, then, suppose $xy \in I$, it follows $xRy \subseteq I$, which implies, either $x\langle s\rangle \subseteq \mathcal{P}(R)$ or $y\langle s\rangle \subseteq I$, consequently, either $sx \in \sqrt{0}$ or $sy \in I$.

In the following we give more examples of S-$\mathcal{P}$-ideals, which are not $\mathcal{P}$-ideals.

Example 4 The ideal $I = M_n(\langle 12X\rangle)$, in Example 3, is an S-$\mathcal{P}$-ideal, but not a $\mathcal{P}$-ideal.

Example 5 Let $R = M_2(\mathbb{Z}_{12})$. Let

$$S_R = \left\{\begin{bmatrix} s & 0 \\ 0 & s\end{bmatrix}; s \in S = \{1, 3, 9\}\right\}, \text{ and } s^* = \begin{bmatrix} 3 & 0 \\ 0 & 3\end{bmatrix}.$$

Then, S_R is an m-system of the ring R. Consider the ideal $P = M_2(\langle 4\rangle)$. Then, $\mathcal{P}(R) = M_n(\langle 6\rangle)$, and $P \cap S_R = \phi$. Since

$$\begin{bmatrix} 2 & 0 \\ 0 & 2\end{bmatrix} R \begin{bmatrix} 2 & 0 \\ 0 & 2\end{bmatrix} \subseteq P \text{ and } \begin{bmatrix} 2 & 0 \\ 0 & 2\end{bmatrix} \notin \mathcal{P}(R), \begin{bmatrix} 2 & 0 \\ 0 & 2\end{bmatrix} \notin P,$$

then, P is not a $\mathcal{P}$-ideal. Now assume $\mathcal{M}_1 R\mathcal{M}_2 \subseteq P$, for some matrices $\mathcal{M}_1, \mathcal{M}_2 \in R$. If $\mathcal{M}_2\langle s^*\rangle \subseteq P$, then, P is an S_R-$\mathcal{P}$-ideal. Thus, assume $\mathcal{M}_2\langle s^*\rangle \nsubseteq P$. Then, $\mathcal{M}_1 R\mathcal{M}_2 \subseteq M_2(\langle 2\rangle) = J$, and since J is prime, then either $\mathcal{M}_1 \in J$ or $\mathcal{M}_2 \in J$, now we discuss two cases as the following.

If $\mathcal{M}_1 \in J$, then, $\mathcal{M}_1\langle s^*\rangle \subseteq \mathcal{P}(R)$, and again P is an S_R-$\mathcal{P}$-ideal.

If $\mathcal{M}_1 \notin J$, then at least one of the entries of $\mathcal{M}_1$ is an odd number and $\mathcal{M}_2 \in J$, hence,

$$\mathcal{M}_2 = \begin{bmatrix} 2l_1 & 2l_2 \\ 2l_3 & 2l_4\end{bmatrix}, \text{ where } l_i \in \mathbb{Z}^+ \text{ for } i \in \{1, 2, 3, 4\}.$$

But by the initial assumption $\mathcal{M}_2\langle s^*\rangle \nsubseteq P$, thus there exists

$$\mathcal{I} = \begin{bmatrix} 3\beta_1 & 3\beta_2 \\ 3\beta_3 & 3\beta_4\end{bmatrix} \in \langle s^*\rangle,$$

where $\beta_i \in \mathbb{Z}^+$, for $i \in \{1, 2, 3, 4\}$, such that $\mathcal{M}_2\mathcal{I} \notin P = M_2(\langle 4\rangle)$, which means that at least one of the entries of $\mathcal{M}_2\mathcal{I}$ is not a multiple of 4 in $\mathbb{Z}_{12}$. Without loss of generality, suppose that $6(l_1\beta_1 + l_2\beta_3)$ is not a multiple of 4, then, either l_1 or/and l_2 is/are odd. Thus, at least one of k_i must be odd for $i \in \{1, 2, 3, 4\}$, consequently, at least one of the even entries of $\mathcal{M}_2$ is not a multiple of 4. By considering all the cases of the positions of the odd entries of $\mathcal{M}_1$ and the non multiple of 4 entries of $\mathcal{M}_2$, we can find, in each case, an element $\mathcal{M}$ of R, such that $\mathcal{M}_1\mathcal{M}\mathcal{M}_2 \notin P$, i.e., $\mathcal{M}_1 R\mathcal{M}_2 \nsubseteq P$, which is a contradiction. That is why $\mathcal{M}_1$ must be an element of J, and thus, $\mathcal{M}_1\langle s^*\rangle \subseteq \mathcal{P}(R)$.

Theorem 14 *Let R be a ring with identity and $I \lhd R$ with $I \cap S = \phi$. Then, for some $s \in S$, $(I : \langle s\rangle)$ is an S-ρ-ideal, if and only if, I is an S-$\mathcal{P}$-ideal, if and only if, I is an S-$\mathcal{P}$-ideal.*

Proof By taking ρ to be the prime radical, the proof follows from Theorem 4. □

Proposition 15 *Let R be a ring with identity and $I \lhd R$ such that $I \cap S = \phi$. If for some $s \in S$, $(I : \langle s\rangle)$ is a $\mathcal{P}$-ideal, then, I is an S-$\mathcal{P}$-ideal.*

Proof Suppose $(I : \langle s\rangle)$ is a $\mathcal{P}$-ideal, and let $xRy \subseteq I$ for some $x, y \in R$, then, $xRy \subseteq (I : \langle s\rangle)$, hence by assumption, either $x \in \mathcal{P}(R)$ or $y \in (I : \langle s\rangle)$. If $x \in \mathcal{P}(R)$, then $x\langle s\rangle \subseteq \mathcal{P}(R)$. If $y \in (I : \langle s\rangle)$, then, $y\langle s\rangle \subseteq I$. Thus, by (3) of Proposition 13, I is an S-$\mathcal{P}$-ideal. □

The following proposition, shows the converse of the above proposition holds when S is contained in the center of the ring R ($C(R)$), and $(\mathcal{P}(R) : \langle s\rangle)$ is a $\mathcal{P}$-ideal, for some $s \in S$.

Proposition 16 *Let P be an ideal of a ring R with identity, S be an m-system of R such that $S \subseteq C(R)$, and $(\mathcal{P}(R) : \langle s\rangle)$ is a $\mathcal{P}$-ideal, disjoint from S, for some $s \in S$. If P is an S-$\mathcal{P}$-ideal, then, $(P : \langle s\rangle)$ is a $\mathcal{P}$-ideal.*

Proof Since $(\mathcal{P}(R) : \langle s\rangle)$ is a $\mathcal{P}$-ideal, then, by Proposition 1.5 of [1], $(\mathcal{P}(R) : \langle s\rangle) \subseteq \mathcal{P}(R)$, and hence, $(\mathcal{P}(R) : \langle s\rangle) = \mathcal{P}(R)$. Let $xRy \subseteq (P : \langle s\rangle)$ for some $x, y \in R$, then, $\langle x\rangle\langle y\rangle\langle s\rangle \subseteq P$, and by assumption, either $\langle x\rangle\langle s\rangle \subseteq \mathcal{P}(R)$ or $\langle y\rangle\langle s\rangle\langle s\rangle \subseteq P$. If $\langle x\rangle\langle s\rangle \subseteq \mathcal{P}(R)$, then, $x \in (\mathcal{P}(R) : \langle s\rangle) = \mathcal{P}(R)$. If $\langle y\rangle\langle s\rangle\langle s\rangle \subseteq P$, then, $\langle s\rangle\langle s\rangle\langle y\rangle \subseteq P$, and again either $\langle s\rangle^3 \subseteq \mathcal{P}(R)$, which implies for some $r \in R$, $s' = srs \in \langle s\rangle^2 \subseteq (\mathcal{P}(R) : \langle s\rangle) = \mathcal{P}(R)$, a contradiction, or $\langle y\rangle\langle s\rangle \subseteq P$, which implies $y \in (P : \langle s\rangle)$. Thus, $(P : \langle s\rangle)$ is a $\mathcal{P}$-ideal. □

In the following, we denote the set of the right zero divisors of a ring R by $Z_r(R)$.

Corollary 11 *Let $Z_r(R) \cap S = \phi$. If $P \cap S = \phi$ for each prime ideal P of R, then, the zero ideal is a $\mathcal{P}$-ideal, if and only if, it is an S-$\mathcal{P}$-ideal.*

Proof Suppose the zero ideal is a $\mathcal{P}$-ideal, then, clearly, it is an S-$\mathcal{P}$-ideal. Conversely, suppose the zero ideal is an S-$\mathcal{P}$-ideal, $xRy = 0$, and $x\langle s\rangle \nsubseteq \mathcal{P}(R)$ for some $x, y \in R$, and all $s \in S$. Then, by assumption, $y\langle s\rangle = 0$. Since $s \notin Z_r(R)$, then, $y = 0$, consequently, the zero ideal is a $\mathcal{P}$-ideal. □

Theorem 17 *Let $f : R_1 \to R_2$ be a ring epimorphism, and P be an S-$\mathcal{P}$-ideal of R_1 such that $Ker(f) \subseteq P$. Then $f(P)$ is an $f(S)$-$\mathcal{P}$-ideal of R_2.*

Proof By taking ρ to be the prime radical, the proof follows from Theorem 5. $\square$

Theorem 18 *Let S be an m-system of R_1. Let $f\colon R_1 \to R_2$ be a ring epimorphism, and P be an ideal of R such that $Ker(f) \subseteq P \cap \mathcal{P}(R)$ and $P \cap S = \phi$. If $f(P)$ is an $f(S)$-$\mathcal{P}$ ideal of R_2, then P is an S-$\mathcal{P}$-ideal of R_1.*

Proof By taking ρ to be the prime radical, the proof follows from Theorem 6. $\square$

Proposition 19 *Let R be a ring with identity, $P \lhd R$ and $(\mathcal{P}(R) : \langle s \rangle) = \mathcal{P}(R)$ for some $s \in S$. Then, the following statements are equivalent.*

(1) *P is an S-$\mathcal{P}$-ideal of R associated with $s \in S$.*
(2) *(i) $P \subseteq (\mathcal{P}(R) : \langle s \rangle)$.*
   (ii) For all $x, y \in R$ with $xRy \subseteq P$, either $x \in (\mathcal{P}^*(P) : \langle s \rangle)$ or $y \in (P : \langle s \rangle)$.

Proof By taking ρ to be the prime radical, the proof follows from Proposition 2. $\square$

Proposition 20 *Let R be ring with identity and $P \lhd R$ such that $S \cap P = \phi$.*

(1) *If P is an S-$\mathcal{P}$-ideal, then, $P \subseteq (\mathcal{P}(R) : \langle s \rangle)$ for some $s \in S$.*
(2) *$\mathcal{P}(R)$ is an S-$\mathcal{P}$-ideal if and only if $\mathcal{P}(R)$ is a right S-prime ideal.*

Proof By taking ρ to be the prime radical, the proof follows from Proposition 3. $\square$

Recall that an ideal P of a ring R is called superfluous if $A + B = R$ for some ideal B of R, then, $B = R$.

Corollary 12 *For a local ring R with unique maximal ideal $\mathcal{M}$, if $(\mathcal{P}(R) : \langle s \rangle)$ is a $\mathcal{P}$-ideal of R, then K is a superfluous ideal for any S-$\mathcal{P}$-ideal K of R.*

Proof Suppose $K + J = R$ for some ideal J of R, then, there exist $i \in K$ and $j \in J$ such that $i + j = 1$, hence, $1 - j = i \in K$. Now since K is an S-$\mathcal{P}$-ideal of R, then, by Proposition 19, $K \subseteq (\mathcal{P}(R) : \langle s \rangle)$, and since $(\mathcal{P}(R) : \langle s \rangle)$ is a $\mathcal{P}$-ideal, then, by Proposition 1.5 of [1], $(\mathcal{P}(R) : \langle s \rangle) = \mathcal{P}(R)$. Consequently,

$$1 - j = i \in K \subseteq (\mathcal{P}(R) : \langle s \rangle) = \mathcal{P}(R) \subseteq \mathcal{J}(R) = \mathcal{M}.$$

Thus, j is a unit, and hence, $J = R$, and K is superfluous. $\square$

Theorem 21 *Let $A, B \lhd R$ such that $A \subseteq B$, and S be an m-system of R. Then, we have the followings:*

(1) *If B is an S-$\mathcal{P}$-ideal of R, then, B/A is an $\overline{S}$-$\mathcal{P}$-ideal of R/A.*
(2) *If B/A is an $\overline{S}$-$\mathcal{P}$-ideal of R/A, and $A \subseteq \mathcal{P}(R)$, then, B is an S-$\mathcal{P}$-ideal of R.*
(3) *If B/A is an $\overline{S}$-$\mathcal{P}$-ideal of R/A, and A is a $\mathcal{P}$-ideal of R, then, B is an S-$\mathcal{P}$-ideal of R.*
(4) *If R_1 is a subring of R_2 and I_2 is an S-$\mathcal{P}$-ideal of R_2, then $I_2 \cap R_1$ is an S-$\mathcal{P}$-ideal of R_1.*

Proof By taking ρ to be the prime radical, the proof follows from Theorem 7. $\square$

Corollary 13 *Let $S \subseteq R$ be an m-system of a ring R. Let I and J be S-$\mathcal{P}$-ideals of R such that $(I + J) \cap S = \phi$, and $I \subseteq \mathcal{P}(R)$. Then, $I + J$ is an S-$\mathcal{P}$-ideal of R.*

Proof Since I is an S-$\mathcal{P}$-ideal of R, then, $I/(J \cap I)$ is an $\bar{S}$-$\mathcal{P}$-ideal of $R/(J \cap I)$ by Theorem 21. Since $(J + I)/I \cong I/(J \cap I)$, then $(J + I)/I$ is an $\bar{S}$-$\mathcal{P}$-ideal of R/I. Thus, by Theorem 21, $J + I$ is an S-$\mathcal{P}$-ideal of R. □

Theorem 22 *Let R be a ring, M an R-R-bimodule, and P an ideal of R. Then, $P \boxplus M$ is an $(S \boxplus M)$-$\mathcal{P}$-ideal of $R \boxplus M$ if and only if P is an S-$\mathcal{P}$-ideal of R.*

Proof By taking ρ to be the prime radical, the proof follows from Theorem 12. □

It is well known that if the rings R_1, R_2 are with identities, then any ideal of $R_1 \times R_2$ has the form $I \times J$, where I and J are ideals of R_1 and R_2, respectively. In addition, $\mathcal{P}(R_1 \times R_2) = \mathcal{P}(R_1) \times \mathcal{P}(R_2)$, see Lemma 3 of [12]. Moreover, if S_1 is an m-system of R_1 and S_2 is an m-system of a ring S_2, then, $S_1 \times S_2$ is an m-system of the ring $R_1 \times R_2$. It is easy to check that for an ideal I of R_1, $(I \times R_2) \cap (S_1 \times S_2) = \phi$, if and only if $I \cap S_1 = \phi$.

Theorem 23 *Let $R = R_1 \times R_2$, where R_1 and R_2 are rings with identities, and let $S = S_1 \times S_2$, where S_1 and S_2 are m-systems of R_1 and R_2, respectively. If P_1 is an ideal of R_1, then, $P_1 \times R_2$ is an S-$\mathcal{P}$-ideal, if and only if, P_1 is an S_1-$\mathcal{P}$-ideal and $\mathcal{P}(R_2) \cap S_2 \neq \phi$.*

Proof By the remark in the above paragraph $(P_1 \times R_2) \cap S = \phi$, if and only if $P_1 \cap S_1 = \phi$. Assume that $P = P_1 \times R_2$ is an S-$\mathcal{P}$-ideal of R. Since $(0, 1)R(1, 0) \subseteq P$, then, either $(0, 1)\langle(s_1, s_2)\rangle \subseteq \mathcal{P}(R_1 \times R_2) = \mathcal{P}(R_1) \times \mathcal{P}(R_2)$ or $(1, 0)\langle(s_1, s_2)\rangle \subseteq P$ for some $(s_1, s_2) \in S$, hence, either $\langle s_2\rangle \subseteq \mathcal{P}(R_2)$, and hence, $\mathcal{P}(R_2) \cap S_2 \neq \phi$, or $\langle s_1\rangle \subseteq P_1$, which is a contradiction. Let $IJ \subseteq P_1$ for some ideals I, J of R_1, then, $(I \times 0)(J \times 0) \subseteq P$, and hence, by assumption, either $(I \times 0)\langle(s_1, s_2)\rangle \subseteq \mathcal{P}(R_1) \times \mathcal{P}(R_2)$ or $(J \times 0)\langle(s_1, s_2)\rangle \subseteq P$, consequently, either $I\langle s_1\rangle \subseteq \mathcal{P}(R_1)$ or $J\langle s_1\rangle \subseteq P_1$. Since $P \cap S = \phi$, then we get $P_1 \cap S_1 = \phi$. Thus, P_1 is an S_1-$\mathcal{P}$-ideal of R_1.

Conversely, suppose P_1 is an S_1-$\mathcal{P}$-ideal and $\mathcal{P}(R_2) \cap S_2 \neq \phi$. Let $(I_1 \times J_1)(I_2 \times J_2) \subseteq P$ for some ideals I_1, I_2 of R_1 and some ideals J_1, J_2 of R_2, then, $(I_1 I_2) \times (J_1 J_2) \subseteq P_1 \times R_2$, hence, $I_1 I_2 \subseteq P_1$, consequently, either $I_1\langle s_1\rangle \subseteq \mathcal{P}(R_1)$ or $I_2\langle s_1\rangle \subseteq P_1$ for some $s_1 \in S_1$. In addition, let $s_2 \in \mathcal{P}(R_2) \cap S_2$, in this case we obtain,

$$\text{either } I_1\langle s_1\rangle \times J_1\langle s_2\rangle \subseteq \mathcal{P}(R_1) \times \mathcal{P}(R_2) \text{ or } I_2\langle s_1\rangle \times J_2\langle s_2\rangle \subseteq P_1 \times R_2.$$

Hence,

$$\text{either } (I_1 \times J_1)\langle(s_1, s_2)\rangle \subseteq \mathcal{P}(R_1 \times R_2) \text{ or } (I_2 \times J_2)\langle(s_1, s_2)\rangle \subseteq P_1 \times R_2.$$

Thus, $P = P_1 \times R_2$ is an S-$\mathcal{P}$-ideal of R. □

Theorem 24 *Let $R = R_1 \times R_2$, where R_1 and R_2 are rings with identities, and let $S = S_1 \times S_2$, where S_1 and S_2 are m-systems of R_1 and R_2, respectively. If P_2 is an ideal of R_2, then, $R_1 \times P_2$ is an S-$\mathcal{P}$-ideal, if and only if, P_2 is an S_2-$\mathcal{P}$-ideal and $\mathcal{P}(R_1) \cap S_1 \neq \phi$.*

Proof Similar to the proof of Theorem 23. □

The condition $\mathcal{P}(R_2) \cap S_2 \neq \phi$ ($\mathcal{P}(R_1) \cap S_1 \neq \phi$) in Theorem 23 (in Theorem 24) is not superfluous, as we show in the following example.

Example 6 Let $R = M_2(\mathbb{Z}_{12})$, and let

$$S_R = \left\{ \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}; s \in S = \{1, 3, 9\} \right\}.$$

Then, we show in Example 5 that the ideal $P = M_2(\langle 4 \rangle)$ is an S_R-$\mathcal{P}$-ideal, and $\mathcal{P}(R) = M_n(\langle 6 \rangle)$. Now in the ring $R' = R \times R$, we have the m-system $S' = S_R \times S_R$, and $\mathcal{P}(R) \cap S_R = \phi$, then the ideal $P \times R$ is not an S'-$\mathcal{P}$-ideal, because, for the matrices:

$$\mathcal{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathcal{M}_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \text{ and } \mathcal{M}_2 = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix},$$

we have $(\mathcal{M}_1, \mathcal{I})R'(\mathcal{M}_2, \mathcal{I}) \subseteq P \times R$, however, $(\mathcal{M}_1, \mathcal{I})\langle(s_1, s_2)\rangle \nsubseteq \mathcal{P}(R')$ and $(\mathcal{M}_2, \mathcal{I})\langle(s_1, s_2)\rangle \nsubseteq P \times R$, for all $(s_1, s_2) \in S'$, because $\mathcal{I}\langle s_2 \rangle \nsubseteq \mathcal{P}(R)$ and $\mathcal{M}_2\langle s_1 \rangle \nsubseteq P$.

Acknowledgements The authors would like to thank the referee for their useful comments and suggestions.

Declarations

Conflict of interest The authors declare no conflict of interest.

References

1. Groenewald, N.: On radical ideals of noncommutative rings. J. Algebr. Appl. **22**(09), 2350196 (2023)
2. Groenewald, N.: On weakly radical ideals of non-commutative rings. J. Algeb. Appl. **22**(12), 2330003 (2023)
3. Tekir, U., Koc, S., Oral, K.H.: n-Ideals of Commutative rings. Filomat **31**(10), 2933–2941 (2017)
4. Khashan, H., Yetkin, E.: S-n-ideals of commutative rings. Commun. Faculty Sci. Univ. Ankara Ser.A1 Math. Stat. **72**(1), 199–215 (2023)
5. Abouhalaka, A.: S-Prime Ideals, S-Noetherian Noncommutative Rings, and the S-Cohen's Theorem. Mediterr. J. Math. **21**(43),(2024)
6. Ersoy, B., Tekir, U., Yıldız, E.: S-versions and S-generalizations of idempotents, pure ideals and stone type theorems. Bull. Korean Math. Soc. **61**, 83–92 (2024). https://doi.org/10.4134/BKMS.b230023
7. Amitsur, S.A.: Radicals of polynomials rings. Canad. J. Math. **8**, 355–361 (1956)
8. Lam, T.Y.: A First Course in Noncommutative Rings. Graduate Texts in Mathematics. Springer, New York (2001)
9. Abouhalaka, A., Fındık, Ş: Generalization of primary ideal associated with a special radical class. Sao Paulo J. Math. Sci. **18**, 1–20 (2023)
10. Andrunakievich, V.A.: Radicals of associative rings I. Amer. Math. Soc. Transl. **2**(52), 95–128 (1966)
11. Veldsman, S.: A note on the radicals of idealizations. Southeast Asian Bull. Math. **32**, 545–551 (2008)

12. Abouhalaka, A., Fındık, Ş: Extension of almost primary ideals to noncommutative rings and the generalization of nilary ideals. Mathematics **11**(8), 1917 (2023)
13. Birkenmeier, G.F., Kim, J.Y., Park, J.K.: Right primary and nilary rings and ideals. J. Algeb. **378**, 133–152 (2013)
14. Gardner, B.J., Wiegandt, R.: Radical Theory of Rings. Marcel Dekker Inc., New York (2004)
15. Khashan, H.A., Bani-Ata, A.B.: J-ideals of commutative rings, international electronic. J. Algeb. **29**, 148–164 (2021)
16. Abouhalaka, A.: A note on weakly semiprime ideals and their relationship to prime radical in non-commutative rings. J. Math. **2024**, 9142090 (2024)
17. Abouhalaka, A.: S-Prime Right Submodules and an S-Version of Prime Avoidance, arXiv preprint arXiv:2401.07270, (2024)
18. Abouhalaka, A., Fındık, Ş: Almost prime ideals in noncommutative rings. Serdica Math. J. Sofia, Bulgaria **48**(4), 235–246 (2022). https://doi.org/10.55630/serdica.2022.48.235-246
19. Baeck, J., Lee, G., Lim, J.W.: S-Noetherian rings and their extensions. Taiwanese J. Math. **20**, 1231–1250 (2016)
20. Baeck, J.: On S-principal right ideal rings. AIMS Math. **7**, 12106–12122 (2022)
21. Reyes, M.L.: Noncommutative generalizations of theorems of cohen and kaplansky. Algebr. Represent Theor. **15**, 933–975 (2012)
22. Lee, G., Baeck, J.: Lim, J: Eakin-Nagata-Eisenbud theorem for right S-Noetherian rings. Taiwanese J. Math. **27**(2), 237–257 (2023)
23. Bilgin, Z., Reyes, M.L., Tekir, U.: On right S-Noetherian rings and S-Noetherian modules. Comm. Algebra **46**(2), 863–869 (2018)