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Research Article

S-$\mathcal{J}$-Ideals: A Study in Commutative and Noncommutative Rings

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In this paper, we introduce the concept of S-$\mathcal{J}$-ideals in both commutative and noncommutative rings. For a commutative ring R and a multiplicatively closed subset S, we show that many properties of $\mathcal{J}$-ideals apply to S-$\mathcal{J}$-ideals and examine their characteristics in various ring constructions, such as homomorphic image rings, quotient rings, cartesian product rings, polynomial rings, power series rings, idealization rings, and amalgamation rings. In noncommutative rings, where S is an m-system, we define right S-$\mathcal{J}$-ideals. We demonstrate the equivalence of S-$\mathcal{J}$-ideals and right S-$\mathcal{J}$-ideals in commutative rings with identity and provide examples to illustrate the connections between right S-prime ideals and $\mathcal{J}$-ideals.

Keywords: commutative ring; $\mathcal{J}$-ideal; m-system; multiplicatively closed subset; noncommutative ring; S-$\mathcal{J}$-ideals

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1. Introduction

Let R be a commutative ring and S be a multiplicatively closed subset of R. The notion of S-prime ideals is first introduced in 2020 in [2]. An ideal I, disjoint from S, of commutative ring R is deemed S-prime if for all $a, b \in R$ with $ab \in I$, it follows that $as \in I$ or $bs \in I$. This idea has been expanded upon in many studies, including [1, 3–10]. The concept of $\mathcal{J}$-ideals first proposed in [11] serves as a broader generalization of n-ideals introduced in [12]. An ideal I of R is termed a $\mathcal{J}$-ideal (an n-ideal) if for all $a, b \in R$ with $ab \in I$, and $a \notin \mathcal{J}(R)$ ($a \notin \beta(R)$), then $b \in I$, where $\mathcal{J}(R)$ and $\beta(R)$ are the Jacobson and prime radicals of R, respectively. Recently, the concept of S-n-ideals has been defined in [13], where an ideal I disjoint from S is called an S-n-ideal if for all $a, b \in R$ with $ab \in I$, and $as \notin \beta(R)$, then $bs \in I$.

In noncommutative rings, the concept of the right S-prime ideal is introduced in [14], where S is an m-system. Definition 2.1 of [14] states that an ideal P disjoint from S of a noncommutative ring R is called right S-prime, if and only if whenever $IK \subseteq P$, then either $I\langle s\rangle \subseteq P$ or $K\langle s\rangle \subseteq P$ for all ideals I, K of R, and for some (fixed) $s \in S$. Just as multiplicatively closed subsets are closely linked to prime ideals in commutative rings, m-systems play a similar role in noncommutative rings. Thus, employing S as an m-system is essential in noncommutative rings to obtain satisfactory results. In addition, the concept of $\mathcal{J}$-ideals in noncommutative case is introduced in [15]. Definition 5.1 of [15] states that a proper ideal I of R is a $\mathcal{J}$-ideal if whenever $a_1, a_2 \in R$ with $a_1 R a_2 \subseteq I$ and $a_1 \notin \mathcal{J}(R)$, then $a_2 \in I$. Recall that the concept of $\mathcal{J}$-ideals is also generalized in [16] based on the earlier work presented in [17].

In this paper, we introduce the notion of S-$\mathcal{J}$-ideals in both commutative and noncommutative rings. The second section explores the characteristics of S-$\mathcal{J}$-ideals in commutative rings, demonstrating how many properties of $\mathcal{J}$-ideals naturally extend within this framework. We also examine S-$\mathcal{J}$-ideals in various related rings, such as homomorphic image rings, quotient rings, cartesian product rings, polynomial rings, power series rings, idealization rings, and amalgamation rings.

The third section is dedicated to right S-$\mathcal{J}$-ideals in noncommutative rings, where S is an m-system. We illustrate the equivalence of the two new concepts when R is commutative with a unit, provide many examples, and clarify the relationships between right S-prime and $\mathcal{J}$-ideals.

2. S-$\mathcal{J}$-Ideals in Commutative Rings

In this section, by R we mean a commutative ring with identity, and S will denote a multiplicatively closed subset of R.

2.1. Characteristics of S-$\mathcal{J}$-Ideals

Definition 1. An ideal I of R disjoint from S is called an S-$\mathcal{J}$-ideal, if there exists an (a fixed) $s \in S$ such that for all $a, b \in R$, if $ab \in I$, then either $sa \in \mathcal{J}(R)$, or $sb \in I$.

Due to the symmetry between a and b, we can see that if I is an S-$\mathcal{J}$-ideal with $as, bs \notin I$, then $as, bs \in \mathcal{J}(R)$.

From the definitions, it is evident that all the $\mathcal{J}$-ideals qualify as S-$\mathcal{J}$-ideals. However, the converse does not hold, as demonstrated by the upcoming example.

Example 1. Let $R = \mathbb{Z}_{36}$. The Jacobson radical of R is $\mathcal{J}(R) = \langle \overline{6} \rangle$. The ideal $I = \langle \overline{4} \rangle$ is not a $\mathcal{J}$-ideal, since $\overline{2}.\overline{2} \in I$ but neither $\overline{2} \in \mathcal{J}(R)$ nor $\overline{2} \in I$. Take $S = \{\overline{1}, \overline{3}, \overline{9}, \overline{27}\}$. It is obvious that S is a multiplicatively closed subset of R and $S \cap I = \varnothing$. Now, choose $s = \overline{3} \in S$ and assume $ab \in I$ for $a, b \in R$.

- If $3b \in I$, then I is S-$\mathcal{J}$.
- If $3b \notin I$, then, $ab \in \langle \overline{4} \rangle \subseteq \langle \overline{2} \rangle$ implies $a \in \langle \overline{2} \rangle$ or $b \in \langle \overline{2} \rangle$. Assume $a \notin \langle \overline{2} \rangle$ and $b \in \langle \overline{2} \rangle$. Then, $b = \overline{4k_1 - 2}$, for $k_1 \in \mathbb{Z}^+$, and since $a \notin \langle \overline{2} \rangle$, then $a = \overline{2k_2 - 1}$, for $k_2 \in \mathbb{Z}^+$. Notice that $ab = (\overline{2k_2 - 1})(\overline{4k_1 - 2}) \notin I$, a contradiction. Hence, a must be even, and thus $3a \in \mathcal{J}(R)$. Therefore, I is S-$\mathcal{J}$.

Proposition 1. *If I is an S-$\mathcal{J}$-ideal of R, associated with an element $s \in S$, then $I \subseteq (\mathcal{J}(R){:}\ s)$.*

Proof 1. Assume I is an S-$\mathcal{J}$-ideal. If $I \subseteq (\mathcal{J}(R){:}\ s)$. Then, there exists $a \in I$ such that $as \notin \mathcal{J}(R)$. However, $a \cdot 1 \in I$, so $s \in I$, which contradicts with $S \cap I = \varnothing$. Hence, I is contained in $(\mathcal{J}(R){:}\ s)$. □

Corollary 1 (Proposition 2.2 of [11]). *If I is a $\mathcal{J}$-ideal of R, then $I \subseteq \mathcal{J}(R)$.*

Proof 2. Follows from Proposition 1 by taking $S = \{1\}$. □

Corollary 2. *In a ring R, we have the following:*

1. *Every S-n-ideal is an S-$\mathcal{J}$-ideal.*
2. *Every n-ideal is a $\mathcal{J}$-ideal (Proposition 2.4 of [11]).*
3. *$\mathcal{J}(R)$ is an S-$\mathcal{J}$-ideal if and only if $\mathcal{J}(R)$ is S-prime.*

Proof 3

1. Follows from the fact that the prime radical is contained in the Jacobson radical
2. Follows from (1) by taking $S = \{1\}$
3. Obvious □

Actually, in (3) of the previous corollary, it was shown that the Jacobson radical $\mathcal{J}(R)$ is an S-$\mathcal{J}$-ideal if and only if $\mathcal{J}(R)$ is an S-prime ideal. Nevertheless, the two concepts are not comparable, as we illustrate in the following example.

Example 2. Let $R = \mathbb{Z}[X]$ and let $I = \langle 9x \rangle$. Considering the multiplicatively closed subset $S = \{3^n{:}\ n \in \mathbb{N} \cup \{0\}\}$. Then, $\mathcal{J}(R) = \mathcal{J}(\mathbb{Z})[X] = 0$. Example 2.3 in [3] shows that I is S-prime. However, since R is an integral domain, then $(\mathcal{J}(R){:}\ s) = (0{:}\ s) = 0$ for all $s \in S$; hence, $I \subseteq (\mathcal{J}(R){:}\ s)$. Thus, by Proposition 1, I is not S-$\mathcal{J}$.

Proposition 2. *An ideal I of R disjoint from S is an S-$\mathcal{J}$-ideal, if and only if, for all ideals A, B of R with $AB \subseteq I$, there exists an (a fixed) $s \in S$ such that either $As \subseteq \mathcal{J}(R)$, or $Bs \subseteq I$.*

Proof 4. Suppose the ideal I is S-$\mathcal{J}$, and let $AB \subseteq I$ for some ideals A, B of R. If $As \subseteq \mathcal{J}(R)$, then there exists $x \in A$ with $xs \notin \mathcal{J}(R)$. Now, for all $y \in B$, $xy \in AB \subseteq I$; hence, $ys \in I$, consequently, $Bs \subseteq I$. Conversely, let $a_1, a_2 \in R$ with $a_1 a_2 \in I$. Then, $\langle a_1 \rangle \langle a_2 \rangle \subseteq I$, and by assumption, $a_1 s \in \langle a_1 \rangle s \subseteq \mathcal{J}(R)$ or $a_2 s \in \langle a_2 \rangle s \subseteq I$. □

Theorem 1. *An ideal I of R disjoint from S is an S-$\mathcal{J}$-ideal, if $(I{:}\ s)$ is a $\mathcal{J}$-ideal of R for some $s \in S$. The converse holds when $\mathcal{J}(R)$ is a $\mathcal{J}$-ideal disjoint from S.*

Proof 5. Suppose $(I{:}\ s)$ is a $\mathcal{J}$-ideal, and let $a_1 a_2 \in I$ for some $a_1, a_2 \in R$. Then, $a_1 a_2 \in (I{:}\ s)$; hence, either $a_1 \in \mathcal{J}(R)$ or $a_2 \in (I{:}\ s)$. Consequently, either $a_1 s \in \mathcal{J}(R)$ or $a_2 s \in I$. Conversely, suppose the ideal I is S-$\mathcal{J}$, and $\mathcal{J}(R)$ is a $\mathcal{J}$-ideal with $\mathcal{J}(R) \cap S = \varnothing$. Let $a_1 a_2 \in (I{:}\ s)$ for some $a_1, a_2 \in R$; then, $a_1 (a_2 s) \in I$. Hence, either $a_1 s \in \mathcal{J}(R)$ or $a_2 s^2 \in I$. If $a_1 s \in \mathcal{J}(R)$, then, $a_1 \in \mathcal{J}(R)$ because $s \notin \mathcal{J}(R)$. If $a_2 s^2 \in I$, then either $s^3 \in S \cap \mathcal{J}(R)$ which is a contradiction or $a_2 s \in I$, so $a_2 \in (I{:}\ s)$. □

Proposition 3. *An ideal I of R disjoint from S is S-$\mathcal{J}$, if and only if, for some $s \in S$, $(I{:}\ a) \subseteq (\mathcal{J}(R){:}\ s)$ for all $a \notin (I{:}\ s)$.*

Proof 6. Suppose the ideal I is S-$\mathcal{J}$, and let $x \in (I{:}\ a)$, then $xa \in I$, and hence, $x \in (\mathcal{J}(R){:}\ s)$. Conversely, let $a_1 a_2 \in I$ for some $a_1, a_2 \in R$. If $a_2 s \notin I$, then, $a_2 \notin (I{:}\ s)$; hence, by assumption, $a_1 \in (I{:}\ a_2) \subseteq (\mathcal{J}(R){:}\ s)$, and hence, $a_1 s \in \mathcal{J}(R)$. □

Proposition 4. *An ideal I of R disjoint from S is S-$\mathcal{J}$, if and only if, for some $s \in S$, $(I\colon b) \subseteq (I\colon s)$ for all $b \notin (\mathcal{J}(R)\colon s)$.*

Proof 7. Suppose the ideal I is S-$\mathcal{J}$, and let $x \in (I\colon b)$, then, $bx \in I$, and hence, $x \in (I\colon s)$. Conversely, let $a_1 a_2 \in I$ for some $a_1, a_2 \in R$. If $a_1 s \notin \mathcal{J}(R)$, then, $a_1 \notin (\mathcal{J}(R)\colon s)$; hence, by assumption, $a_2 \in (I\colon a_1) \subseteq (I\colon s)$. Thus, $a_2 s \in I$. □

Corollary 3. *(Proposition 2.10 of [11]) Let I be an ideal of R. The following are equivalent.*

1. *I is a $\mathcal{J}$-ideal of R*
2. *$I = (I\colon b)$ for all $b \notin \mathcal{J}(R)$*
3. *For ideals A and B of R, $AB \subseteq I$ and $A \subseteq \mathcal{J}(R)$ we have $B \subseteq I$*
4. *$(I\colon a) \subseteq \mathcal{J}(R)$ for all $a \notin I$*

Proof 8. Consider $S = \{1\}$ in Propositions 3 and 4. Then, $(I\colon s) = I$ and $(\mathcal{J}(R)\colon s) = \mathcal{J}(R)$. Hence,

- $(1) \Rightarrow (2)$: by Proposition 4, $(I\colon b) \subseteq (I\colon s) = I \subseteq (I\colon b)$
- $(2) \Rightarrow (3)$: by Proposition 2 and 4
- $(3) \Rightarrow (1)$: by Proposition 4
- $(3) \Rightarrow (4)$ and $(4) \Rightarrow (3)$: follows from Proposition 3 □

If I is an ideal within R, then, considering I its own ring, Example 4.7 of [18] shows that the intersection of $\mathcal{J}(R)$ and I is precisely $\mathcal{J}(I)$, the Jacobson radical of the ring I.

Corollary 4. *Let I be an S-$\mathcal{J}$-ideal of R, and let P be an ideal of the ring I such that $P = (P\colon i)$ for all $i \in I$. Then, P is an S-$\mathcal{J}$-ideal of the ring I.*

Proof 9. Let $a_1 a_2 \in P$ for some $a_1, a_2 \in I$. Then, $a_1 a_2 \in I$, and hence, $a_1 s \in \mathcal{J}(R)$ or $a_2 s \in I$. If $a_1 s \in \mathcal{J}(R)$, then, $a_1 s \in \mathcal{J}(R) \cap I = \mathcal{J}(I)$. If $a_2 s \in I$, then, $a_1 a_2 s \in I \cap P = P$, and hence, $a_2 s \in (P\colon a_1) = P$. □

An ideal P of a ring R is termed Jacobson if it can be represented as the intersection of maximal ideals of R. While an ideal K is called S-finite (in the sense of [19]) if there exists a finitely generated ideal F such that $Ks \subseteq F \subseteq K$ for some $s \in S$.

Corollary 5. *Let I be an intersection of maximal ideals of R. If I is a finitely generated S-$\mathcal{J}$-ideal, then, $\mathcal{J}(R)$ is S-finite. Particularly, if a Jacobson ideal is finitely generated, then, $\mathcal{J}(R)$ is S-finite.*

Proof 10. Since I is an intersection of maximal ideals, then, $\mathcal{J}(R) \subseteq I$, and since I is S-$\mathcal{J}$, then by Proposition 1, $\mathcal{J}(R) \subseteq I \subseteq (\mathcal{J}(R)\colon s)$ for some $s \in S$. Thus, $\mathcal{J}(R)s \subseteq Is \subseteq \mathcal{J}(R)$, and since Is is finitely generated, then, $\mathcal{J}(R)$ is S-finite. □

Similar to the proof of the above corollary, one can show that for a maximal finitely generated ideal M, if M is S-$\mathcal{J}$, then, $\mathcal{J}(R)$ is S-finite.

Proposition 5. *Let I, J, and A be ideals of R such that $A \subseteq (\mathcal{J}(R)\colon s)$ for all $s \in S$. Then, we have*

1. *If I and J are S-$\mathcal{J}$-ideals, I (J) is finitely generated, and $AI = AJ$, then J (I) is S-finite*
2. *If AI is finitely generated S-$\mathcal{J}$-ideal, then I is S-finite*

Proof 11

1. Suppose I is finitely generated. Since $AI \subseteq J$, then $Is_1 \subseteq J$ for some $s_1 \in S$. Similarly, there exists $s_2 \in S$ such that $Js_2 \subseteq I$. Hence, $Js_2 s_1 \subseteq Is_1 \subseteq J$. For $s = s_1 s_2 \in S$, $Js \subseteq K \subseteq J$, where $K = Is_1$ is finitely generated. Thus, J is S-finite.
2. Since $AI \subseteq AI$, then $Is \subseteq AI \subseteq I$. □

Corollary 6. *(Proposition 2.21 of [11]). Let I, J, and A be ideals of R such that $A \subseteq \mathcal{J}(R)$. Then, the following hold.*

1. *If I and J are $\mathcal{J}$-ideals of R and $AI = AJ$, then $I = J$*
2. *If I is an ideal such that IA is an $\mathcal{J}$-ideal, then $AI = I$*

Proof 12. Follows from Proposition 5 by taking $S = \{1\}$. □

Lemma 1. *Let $\varnothing \neq X$ be a subset of R. If I is an S-$\mathcal{J}$-ideal of R and $X \subseteq I$, then $(I\colon X)$ is an S-$\mathcal{J}$-ideal of R.*

Proof 13. Let $A_1 A_2 \subseteq (I\colon X)$ for some ideals A_1, A_2 of R. Then, $A_1(A_2 X) \subseteq I$, (notice that $A_2 X$ is an ideal). If $A_1 s \subseteq \mathcal{J}(R)$ for all $s \in S$, then $(A_2 X)s \subseteq I$; consequently, $A_2 s \subseteq (I\colon X)$. □

Proposition 6. *In a ring R, we have*

1. *An ideal I that is maximal with respect to being an S-$\mathcal{J}$-ideal is prime*
2. *If a prime ideal I (disjoint from S) have the property $I = (\mathcal{J}(R)\colon s)$ for some $s \in S$, then I is maximal with respect to being an S-$\mathcal{J}$-ideal.*

Proof 14

1. Let $a_1 a_2 \in I$ for some $a_1, a_2 \in R$. If $a_1 \notin I$, then, by Lemma 1, $(I\colon a_1)$ is S-$\mathcal{J}$ containing I, and by the maximality of I, $(I\colon a_1) = I$; hence, $a_2 \in I$.
2. Let $a_1 a_2 \in I$ for some $a_1, a_2 \in R$. Since I is prime, then either $a_1 \in I = (\mathcal{J}(R)\colon s)$, and hence, $a_1 s \in \mathcal{J}(R)$, or $a_2 \in I$, and thus $a_2 s \in I$. Now, let J be any S-$\mathcal{J}$-ideal. Then, by Proposition 1, $J \subseteq (\mathcal{J}(R)\colon s) = I$, which proves the maximality of I. □

It is worth noting that in (2) of the above proposition, the inclusion $I \subseteq (\mathcal{J}(R)\colon s)$ was sufficient to show that the prime ideal I is S-$\mathcal{J}$. Consequently, the converse of Proposition 1 is valid with the added condition that I is prime. We will now present another case where the converse of Proposition 1 also applies. Recall that $\mathcal{J}^*(I)$ represents the Jacobson radical of the ideal I, defined as the intersection of all maximal ideals containing I. It is clear that $\mathcal{J}(R) \subseteq \mathcal{J}^*(I)$.

Proposition 7. *Let I be an ideal of R disjoint from S, and let $(\mathcal{J}(R)\colon s) = \mathcal{J}(R)$ for some $s \in S$. Then, I is S-$\mathcal{J}$ associated with $s \in S$, if and only if for all $a_1, a_2 \in R$ with $a_1a_2 \in I$, either $a_1 s \in \mathcal{J}^*(I)$ or $a_2 s \in I$, and $I \subseteq (\mathcal{J}(R)\colon s)$.*

Proof 15. Suppose I is an S-$\mathcal{J}$-ideal associated with $s \in S$. Then, by Proposition 1, $I \subseteq (\mathcal{J}(R)\colon s)$. Now, let $a_1a_2 \in I$ for some $a_1, a_2 \in R$, then by assumption, either $a_1 \in (\mathcal{J}(R)\colon s) \subseteq (\mathcal{J}^*(I)\colon s)$ or $a_2 \in (I\colon s)$ for some $s \in S$. Conversely, suppose $a_1a_2 \in I$ for some $a_1, a_2 \in R$. Then, $a_1 s \in \mathcal{J}^*(I)$ or $a_2 s \in I$. Now, since $I \subseteq (\mathcal{J}(R)\colon s) = \mathcal{J}(R)$, then $\mathcal{J}^*(I) \subseteq \mathcal{J}(\mathcal{J}(R)) = \mathcal{J}(R)$. Thus, either $a_1 s \in \mathcal{J}(R)$ or $a_2 s \in I$. Hence, I is an S-$\mathcal{J}$-ideal of R. □

Corollary 7. *(Proposition 2.13 of [11]). An ideal I that is maximal with respect to being a $\mathcal{J}$-ideal is prime. If, in particular, $I = \mathcal{J}(R)$, then the converse is true.*

Proof 16. Follows by Proposition 6 by taking $S = \{1\}$. □

2.2. S-$\mathcal{J}$-Ideals in Related Rings. Here, we explore the connections between S-$\mathcal{J}$-ideals of a ring R and those in various related rings, such as homomorphic image rings, quotient rings, cartesian product rings, polynomial rings, power series rings, idealization rings, and amalgamation rings.

Theorem 2. *Let $\psi\colon R_1 \longrightarrow R_2$ be a ring epimorphism and S be a multiplicatively closed subset of R_1. Then, the following hold.*

1. *If P is an S-$\mathcal{J}$-ideal of R_1 with $\operatorname{Ker}(\psi) \subseteq P$, then $\psi(P)$ is an $\psi(S)$-$\mathcal{J}$-ideal of R_2*
2. *If L is an $\psi(S)$-$\mathcal{J}$-ideal of R_2 with $\operatorname{Ker}(\psi) \subseteq \mathcal{J}(R_1)$, then $\psi^{-1}(L)$ is an S-$\mathcal{J}$-ideal of R_1*

Proof 17. First, we need to show that $\psi(P) \cap \psi(S) = \varnothing$. Otherwise, there exists $x \in \psi(P) \cap \psi(S)$ that implies $x = \psi(a) = \psi(s)$ for some $a \in P$ and $s \in S$. Hence $a - s \in \operatorname{Ker}(\psi) \subseteq P$ and $s \in P$, a contradiction.

1. Let $u, v \in R_2$ and $uv \in \psi(P)$. Since ψ is an epimorphism, $u = \psi(a)$ and $v = \psi(b)$ for some $a, b \in R_1$. Since $\psi(a)\psi(b) \in \psi(P)$ and $\operatorname{Ker}(\psi) \subseteq P$, we get $ab \in P$ and thus there is an $s \in S$ such that $sa \in \mathcal{J}(R_1)$ or $sb \in P$. Hence, $\psi(s)u \in \psi(\mathcal{J}(R_1)) \subseteq \mathcal{J}(R_2)$ or $\psi(s)v \in \psi(P)$.
2. Let $x, y \in R_1$ with $xy \in \psi^{-1}(L)$, and assume $xs \notin \mathcal{J}(R_1)$ for all $s \in S$. Then, $\psi(xy) = \psi(x)\psi(y) \in L$ and since L is an $\psi(S)$-$\mathcal{J}$-ideal of R_2, there exists $\psi(s) \in \psi(S)$ such that $\psi(s)\psi(x) \in \mathcal{J}(R_2)$ or $\psi(s)\psi(y) \in L$. If $\psi(s)\psi(x) = \psi(xs) \in \mathcal{J}(R_2)$, then, $\psi(xs)$ is contained in each maximal ideal of R_2, and for any maximal ideal $\mathcal{M}_1$ of R_1, $\psi(\mathcal{M}_1)$ is also maximal of R_2, and $\operatorname{Ker}(\psi) \subseteq \mathcal{J}(R_1) \subseteq \mathcal{M}_1$; hence, $xs \in \psi^{-1}(\psi(xs)) \subseteq \psi^{-1}(\psi(\mathcal{M}_1)) = \mathcal{M}_1$; consequently, $xs \in \mathcal{J}(R_1)$, a contradiction. Thus, $\psi(ys) \in L$, and hence, $ys \in \psi^{-1}(L)$. □

Proposition 8. *Let $P_1, P_2 \lhd R$ with $P_1 \subseteq P_2$, and $\overline{S} = \{s + P_1\colon s \in S\}$. Then, the following hold.*

1. *If P_2 is an S-$\mathcal{J}$-ideal of R_1, then P_2/P_1 is an $\overline{S}$-$\mathcal{J}$-ideal of R/P_1*
2. *If P_2/P_1 is an $\overline{S}$-$\mathcal{J}$-ideal of R/P_1 and $P_1 \subseteq \mathcal{J}(R)$, then P_2 is an S-$\mathcal{J}$-ideal of R*
3. *If P_2/P_1 is an $\overline{S}$-$\mathcal{J}$-ideal of R/P_1 and P_1 is a $\mathcal{J}$-ideal of R, then P_2 is an S-$\mathcal{J}$-ideal of R*

Proof 18

1. Suppose P_2 is an S-$\mathcal{J}$-ideal. Let $\psi\colon R \longrightarrow R/P_1$ be the natural epimorphism defined by $\psi(r) = r + P_1$, for $r \in R$. Recall that $\operatorname{Ker}(\psi) = P_1 \subseteq P_2$. Hence, by Theorem 2 (1), we get $\psi(P_2) = P_2/P_1$ is an $\overline{S}$-$\mathcal{J}$-ideal of R/P_1.
2. By considering the natural epimorphism $\psi\colon R \longrightarrow R/P_1$, since $P_1 \subseteq \mathcal{J}(R)$, then by using Theorem 2 (2), $P_2 = \psi^{-1}(P_2/P_1)$ is an S-$\mathcal{J}$-ideal of R.
3. By (2) and Corollary 1. □

Corollary 8. *Let $\{P_i\colon i \in \Delta\}$ be a nonempty family of S-$\mathcal{J}$-ideal of R. Then, $\cap_{i\in\Delta} P_i$ is an S-$\mathcal{J}$-ideal of R.*

Proof 19. It is clear that $S \cap (\cap_{i\in\Delta} P_i) = \varnothing$. Let $x, y \in R$, such that $yx \in \cap_{i\in\Delta} P_i$ and $xs \notin \mathcal{J}(R)$ for some $s \in S$. Then, $xy \in P_i$ for each $i \in \Delta$. Since P_i is an S-$\mathcal{J}$-ideal for all $i \in \Delta$, we have $ys \in P_i$ for each $i \in \Delta$ and so $ys \in \cap_{i\in\Delta} P_i$. Hence, $\cap_{i\in\Delta} P_i$ is an S-$\mathcal{J}$-ideal of R. □

Referring to [11], Proposition 2.30 states that $R_1 \times R_2$ has no $\mathcal{J}$-ideal for any rings R_1 and R_2. However, the following theorem shows that $R_1 \times R_2$ can have an S-$\mathcal{J}$-ideal if at least one of the rings R_1 or R_2 possesses such an ideal.

Theorem 3. *Let I_1 and I_2 be ideals of R_1 and R_2, respectively. For any multiplicatively closed subsets S_1 and S_2 of R_1 and R_2, respectively, we have the following:*

1. *$I_1 \times R_2$ is an $(S_1 \times S_2)$-$\mathcal{J}$-ideal of $R_1 \times R_2$, if and only if, I_1 is an S_1-$\mathcal{J}$-ideal and $\mathcal{J}(R_2) \cap S_2 \neq \varnothing$*
2. *$R_1 \times I_2$ is an $(S_1 \times S_2)$-$\mathcal{J}$-ideal of $R_1 \times R_2$, if and only if, I_2 is an S_2-$\mathcal{J}$-ideal and $\mathcal{J}(R_1) \cap S_1 \neq \varnothing$*

Proof 20

1. Let $R = R_1 \times R_2$, and $S = S_1 \times S_2$. It is clear that $(I_1 \times R_2) \cap S = \varnothing$, if and only if $I_1 \cap S_1 = \varnothing$. Suppose $I = I_1 \times R_2$ is an S-$\mathcal{J}$-ideal of R, since $(0,1)(1,0) \in I$,

then either $(0,1)(s_1,s_2) \in \mathcal{J}(R_1 \times R_2) = \mathcal{J}(R_1) \times \mathcal{J}(R_2)$ or $(1,0)(s_1,s_2) \in I$ for some $(s_1,s_2) \in S$; hence, either $s_2 \in \mathcal{J}(R_2)$, and hence, $\mathcal{J}(R_2) \cap S_2 \neq \varnothing$, or $s_1 \in I_1$, which is a contradiction. Now, let $a_1 a_1' \in I_1$ for some $a_1, a_1' \in R_1$; then, $(a_1,1)(a_1',1) \in I$. Hence, $(a_1,1)(s_1,s_2) \in \mathcal{J}(R_1 \times R_2) = \mathcal{J}(R_1) \times \mathcal{J}(R_2)$ or $(a_1',1)(s_1,s_2) \in I$ for some $(s_1,s_2) \in S$; consequently, either $a_1 s_1 \in \mathcal{J}(R_1)$, or $a_1' s_1 \in I_1$. Thus, the ideal I_1 is S_1-$\mathcal{J}$ of R_1. Conversely, suppose the ideal I_1 is S_1-$\mathcal{J}$ and $\mathcal{J}(R_2) \cap S_2 \neq \varnothing$. Let $(a_1,b_2)(a_1',b_2') \in I$ for some $(a_1,b_2), (a_1',b_2')$ of R, then $a_1 a_1' \in I_1$, and thus, $a_1 s_1 \in \mathcal{J}(R_1)$ or $a_1' s_1 \in I_1$ for some $s_1 \in S_1$. For any $s_2 \in \mathcal{J}(R_2) \cap S_2$, we have $(a_1,b_2)(s_1,s_2) \in \mathcal{J}(R_1 \times R_2) = \mathcal{J}(R_1) \times \mathcal{J}(R_2)$ or $(a_1',b_2')(s_1,s_2) \in I$. Thus, $I = I_1 \times R_2$ is an S-$\mathcal{J}$-ideal of R.

2. Similar to (1). □

In Theorem 3, the conditions $\mathcal{J}(R_2) \cap S_2 \neq \varnothing$ and $\mathcal{J}(R_1) \cap S_1 \neq \varnothing$ cannot be omitted. Indeed, by taking $R_1 = R_2 = \mathbb{Z}_{36}$, $S_1 = S_2 = \{\overline{1}, \overline{3}, \overline{9}, \overline{27}\}$, and $P = \langle \overline{4} \rangle$, we know from Example 1 that P is an S-$\mathcal{J}$-ideal of R_1 but $P \times R_1$ is not an $(S_1 \times S_2)$-$\mathcal{J}$-ideal of $R_1 \times R_2$ since $(\overline{2},\overline{1})(\overline{2},\overline{1}) \in P \times R_1$ but for each $(s_1,s_2) \in S_1 \times S_2$, neither $(s_1,s_2)(\overline{2},\overline{1}) \in P \times R_1$ nor $(s_1,s_2)(\overline{2},\overline{1}) \in \mathcal{J}(R)$.

In the following, we present another example of S-$\mathcal{J}$-ideal which is not a $\mathcal{J}$-ideal, by using Theorem 3.

Example 3. Let $R = \mathbb{Z}_{36} \times \mathbb{Z}_8$. The Jacobson radical of R is $\mathcal{J}(R) = \langle \overline{6} \rangle \times \langle \overline{2} \rangle$. Take the multiplicatively closed subset: $S_1 = \{1,3,9,27\}$ and let $X = \{0,2,4\}$. Then, $(S_1 \times X)$ is a multiplicatively closed subset (without zero) of R. Example 1 shows that the ideal $I = \langle \overline{4} \rangle$ of $\mathbb{Z}_{36}$ is an S_1-$\mathcal{J}$ ideal. Thus, by Theorem 3, we find that the ideal $I \times \mathbb{Z}_8$ is an $(S_1 \times X)$-$\mathcal{J}$-ideal of R. However, $I \times \mathbb{Z}_8$ is not a $\mathcal{J}$-ideal due to Proposition 2.30 of [11].

In referring to [20], the Jacobson radical of the power series ring $R[|x|]$, where R is commutative with identity, is $\mathcal{J}(R[|x|]) = \mathcal{J}(R) + xR[|x|]$. While Exercises 1 (4) of [21], and Theorem 3 of [22], show that the Jacobson radical of the polynomial ring $R[x]$ is $\mathcal{J}(R[x]) = \mathcal{J}(R)[x]$.

Theorem 4. *Let R be a ring in which $\mathcal{J}(R)$ is a $\mathcal{J}$-ideal. An ideal I of R, disjoint from S is S-$\mathcal{J}$, if and only if $I[|x|]$ is an S-$\mathcal{J}$-ideal of $R[|x|]$.*

Proof 21. Suppose $I[|x|]$ is an S-$\mathcal{J}$-ideal of $R[|x|]$, and let $a_1 a_2 \in I$ for some $a_1, a_2 \in R$. Then, $a_1 a_2 \in I[|x|]$. Hence, either $a_1 s \in \mathcal{J}(R[|x|])$ or $a_2 s \in I[|x|]$ for some $s \in S$, consequently, either $a_1 s \in \mathcal{J}(R)$ or $a_2 s \in I$. Thus, I is S-$\mathcal{J}$. Conversely, suppose the ideal I is S-$\mathcal{J}$, and let $f_1(x) f_2(x) \in I[|x|]$ for some $f_1(x), f_2(x) \in R[|x|]$. Since $\mathcal{J}(R)$ is a $\mathcal{J}$-ideal; then, by Theorem 1, $(I\colon s)$ is a $\mathcal{J}$-ideal, and since $f_1(x) f_2(x) \in I[|x|] \subseteq (I\colon s)[|x|]$, then either $f_1(x) \in \mathcal{J}(R[|x|])$ or $f_2(x) \in (I\colon s)[|x|]$. If $f_1(x) \in \mathcal{J}(R[|x|])$, then $f_1(0) \in \mathcal{J}(R)$; hence, $f_1(0) s \in \mathcal{J}(R)$, and thus, $f_1(x) s \in \mathcal{J}(R) + xR[|x|] = \mathcal{J}(R[|x|])$. If $f_2(x) \in (I\colon s)[|x|] = (I[|x|]\colon s)$, then $f_2(x) s \in I[|x|]$. Thus, $I[|x|]$ is an S-$\mathcal{J}$-ideal of $R[|x|]$. □

Theorem 5. *Let R be a ring in which $\mathcal{J}(R)$ is a $\mathcal{J}$-ideal. An ideal I of R, disjoint from S is S-$\mathcal{J}$, if and only if $I[x]$ is an S-$\mathcal{J}$-ideal of $R[x]$.*

Proof 22. Similar to the proof of the above Theorem 4. □

In the following, we establish the relationship between S-$\mathcal{J}$-ideals of a ring R and those of the idealization ring $R \boxplus M = \{(a,m)\colon a \in R, m \in M\}$, where M is an R-module, which is defined with the usual addition and the multiplication defined as $(a_1,m_1)(a_2,m_2) = (a_1 a_2, a_1 m_2 + a_2 m_1)$, for all $(a_1,m_1), (a_2,m_2) \in R \boxplus M$. If S is a multiplicatively closed subset of R, then $S_M = S \boxplus M$ is a multiplicatively closed subset of $R \boxplus M$. In addition, $\mathcal{J}(R \boxplus M) = \mathcal{J}(R) \boxplus M$.

Theorem 6. *Let I be an ideal of R disjoint from S and M be an R-module. Then, $I \boxplus M$ is an S_M-$\mathcal{J}$-ideal of $R \boxplus M$ if and only if I is an S-$\mathcal{J}$-ideal of R.*

Proof 23. Suppose I is an S-$\mathcal{J}$-ideal of R. Notice that $S_M \cap (I \boxplus M) = \varnothing$. Let $(a_1,m_1)(a_2,m_2) \in I \boxplus M$ for some $a_1, a_2 \in R$, and $m_1, m_2 \in M$. Then, $a_1 a_2 \in I$, and hence, either $a_1 s \in \mathcal{J}(R)$ or $a_2 s \in I$ for some $s \in S$. If $a_2 s \in I$, then $(a_2,m_2)(s,0) \in I \boxplus M$. If $a_1 s \in \mathcal{J}(R)$, then $(a_1,m_1)(s,0) \in \mathcal{J}(R) \boxplus M = \mathcal{J}(R \boxplus M)$, where $(s,0) \in S_M$. Thus, $I \boxplus M$ is an (S_M)-$\mathcal{J}$-ideal of $R \boxplus M$. □

Conversely, suppose $I \boxplus M$ is an (S_M)-$\mathcal{J}$-ideal of $R \boxplus M$. Let $ab \in I$, for some $a, b \in R$. Then, $(a,m_1)(b,m_2) \in I \boxplus M$ for some $m_1, m_2 \in M$, and hence, either $(a,m_1)(s,m) \in \mathcal{J}(R \boxplus M)$ or $(b,m_2)(s,m) \in I \boxplus M$, for some $(s,m) \in S_M$. Consequently, either $as \in \mathcal{J}(R)$ or $bs \in I$. Thus, I is an S-$\mathcal{J}$-ideal of R.

If I is an ideal of R, then for a submodule N of M, $I \boxplus N$ constitutes an ideal of $R \boxplus M$ if and only if $IM \subseteq N$.

Proposition 9. *Let N be a submodule of an R-module M and I be an ideal of R where $IM \subseteq N$. If $I \boxplus N$ is an $S \boxplus M$-$\mathcal{J}$-ideal of $R \boxplus M$, then I is an S-$\mathcal{J}$-ideal of R.*

Proof 24. Since $S_M \cap (I \boxplus N) = \varnothing$, then $S \cap I = \varnothing$. Let $xy \in I$ for some $x, y \in R$. Then, $(x,0)(y,0) \in I \boxplus N$; thus $(x,0)(s,n) \in \mathcal{J}(R) \boxplus M$ or $(y,0)(s,n) \in I \boxplus N$ for some $(s,n) \in S_M$. Thus, $xs \in \mathcal{J}(R)$ or $ys \in I$ and thus I is an S-$\mathcal{J}$-ideal of R. □

The converse of Proposition 9 is not true in general as we show next.

Example 4. In $R = \mathbb{Z}$, the zero ideal is prime and $\mathcal{J}(\mathbb{Z}) = 0$; hence, $(\mathcal{J}(\mathbb{Z})\colon s) = (0\colon s) = 0$ for any $s \in S = \{1,3,9,27,\ldots\}$ because $\mathbb{Z}$ is an integral domain. Thus, the paragraph above Proposition 7, the zero ideal is S-$\mathcal{J}$. Take the $\mathbb{Z}$-module $\mathbb{Z}_6$, then $0 \boxplus 0$ is not an $(S \boxplus \mathbb{Z}_6)$-$\mathcal{J}$-ideal because $(2,0)(0,3)$

$\in 0 \boxplus 0$; however, $(2s, 2m) = (s, m)(2, 0) \notin \mathcal{J}(\mathbb{Z}) \boxplus Z_6 = 0 \boxplus Z_6$ and $(0, 3s) = (s, m)(0, 3) \notin 0 \boxplus 0$ for all $(s, m) \in S \boxplus Z_6$.

Let R and A be two commutative rings, $f\colon R \longrightarrow A$ be a ring homomorphism, and J be an ideal of the ring A. The amalgamation of R with A along J with respect to f, denoted by $R \bowtie^f J$, is the subring of $R \times A$ given by

$$R \bowtie^f J = \{r, f(r) + j \mid r \in R, j \in J\}. \tag{1}$$

This construction is a generalization of the amalgamated duplication of a ring along an ideal. Recall that for an ideal J of a ring R, the amalgamated duplication of R along J, denoted by $R \bowtie J$, is the subring of $R \times R$ given by

$$R \bowtie J = \{r, r + j \mid r \in R, j \in J\}. \tag{2}$$

In addition, if I is an ideal of R, then $I \bowtie^f J = \{i, f(i) + j \mid i \in I, j \in J\}$ is an ideal of $R \bowtie^f J$. Moreover, for a multiplicatively closed subset S of R, $S^{\bowtie} = \{s, f(s) \mid s \in S\}$ is a multiplicatively closed subset of $R \bowtie^f J$. In addition, by Lemma 2.15 of [23], we have $\mathcal{J}(R \bowtie^f J) = \{(r, f(r) + j)\colon r \in \mathcal{J}(R), f(r) + j \in \mathcal{J}\ \ (f(R) + J)\}$. While by Lemma 3 of [24], if $J \subseteq \mathcal{J}(A)$, then $\mathcal{J}(R \bowtie^f J) = \mathcal{J}(R) \bowtie^f J$.

Proposition 10. *Let $f\colon R \longrightarrow A$ be a ring homomorphism, J be an ideal of the ring A, and $R \bowtie^f J$ be the amalgamation of R with A along J with respect to f. Let I be an ideal of R disjoint from S.*

1. *If $I \bowtie^f J$ is an $S^{\bowtie}$-$\mathcal{J}$-ideal of $R \bowtie^f J$, then, I is an S-$\mathcal{J}$-ideal of R*
2. *If I is an S-$\mathcal{J}$-ideal of R, and $J \subseteq \mathcal{J}(A)$, then, $I \bowtie^f J$ is an $S^{\bowtie}$-$\mathcal{J}$-ideal of $R \bowtie^f J$*

Proof 25. Since $S \cap I = \varnothing$, then, $S^{\bowtie} \cap (I \bowtie^f J) = \varnothing$.

1. Suppose that $I \bowtie^f J$ is an $S^{\bowtie}$-$\mathcal{J}$-ideal of $R \bowtie^f J$, and $ab \in I$ for some $a, b \in R$. Then, $(a, f(a))(b, f(b)) \in I \bowtie^f J$, and hence by assumption, either $(a, f(a))(s, f(s)) \in \mathcal{J}(R \bowtie^f J)$ or $(b, f(b))(s, f(s)) \in I \bowtie^f J$ for some $(s, f(s)) \in S^{\bowtie}$. If $(a, f(a))(s, f(s)) \in \mathcal{J}(R \bowtie^f J)$, then $as \in \mathcal{J}(R)$. If $(b, f(b))(s, f(s)) \in I \bowtie^f J$ then, $bs \in I$.
2. Suppose that I is an S-$\mathcal{J}$-ideal of R, and that

$$(a, f(a) + j_1)(b, f(b) + j_2) \in I \bowtie^f J, \tag{3}$$

for some $a, b \in R$ and $j_1, j_2 \in J$. Then, $ab \in I$, and hence by assumption, either $as \in \mathcal{J}(R)$ or $bs \in I$. If $bs \in I$, then $(b, f(b) + j_2)(s, f(s)) \in I \bowtie^f J$. If $as \in \mathcal{J}(R)$, then $(a, f(a) + j_1)(s, f(s)) \in \mathcal{J}(R) \bowtie^f J = \mathcal{J}(R \bowtie^f J)$. □

3. S-$\mathcal{J}$-Ideals in Noncommutative Rings

In this section, unless otherwise specified, R will refer to an associative, noncommutative ring without identity. Also, S will denote an m-system of R. We now present our definition of a right S-$\mathcal{J}$-ideal, in line with Definition 2.2 of [14].

Definition 2. An ideal P of R, that is disjoint from S, is called a right S-$\mathcal{J}$-ideal if and only if whenever $IK \subseteq P$; then, either $I\langle s\rangle \subseteq \mathcal{J}(R)$ or $K\langle s\rangle \subseteq P$ for all ideals I, K of R and for some (fixed) $s \in S$.

Definition 5.1 of [15] states that an ideal P of R is a $\mathcal{J}$-ideal if whenever $a_1, a_2 \in R$ with $a_1 R a_2 \in P$, then $a_1 \in \mathcal{J}(R)$ or $a_2 \in P$. Equivalently, for ideals I, K of R, with $IK \subseteq P$ either $I \subseteq \mathcal{J}(R)$ or $K \subseteq P$. It is easy to check that every $\mathcal{J}$-ideal, disjoint from S is a right S-$\mathcal{J}$-ideal, and the classes of $\mathcal{J}$-ideals and right S-$\mathcal{J}$-ideals coincide when $S \subseteq U(R)$.

Proposition 11. *Let R be a ring with an identity and $P \lhd R$ with $P \cap S = \varnothing$. The following are equivalent:*

1. *P is a right S-$\mathcal{J}$-ideal*
2. *For all $x, y \in R$ with $\langle x\rangle\langle y\rangle \subseteq P$, either $\langle x\rangle\langle s\rangle \subseteq \mathcal{J}(R)$ or $\langle y\rangle\langle s\rangle \subseteq P$ for some $s \in S$*
3. *For all $x, y \in R$ with $xRy \subseteq P$, either $x\langle s\rangle \subseteq \mathcal{J}(R)$ or $y\langle s\rangle \subseteq P$ for some $s \in S$*

Proof 26. $(1) \Rightarrow (2)$ Clear by Definition 2.

$(2) \Rightarrow (3)$ Let $xRy \subseteq P$ for some $x, y \in R$. Then, $\langle x\rangle\langle y\rangle \subseteq P$; hence, by (2), either $x\langle s\rangle \subseteq \langle x\rangle\langle s\rangle \subseteq \mathcal{J}(R)$, or $y\langle s\rangle \subseteq \langle y\rangle\langle s\rangle \subseteq P$ for some $s \in S$.

$(3) \Rightarrow (1)$ Suppose $IJ \subseteq P$, for some ideals I, J of R. If $I\langle s\rangle \subseteq \mathcal{J}(R)$, then there exists $a \in I$ such that $a\langle s\rangle \subseteq \mathcal{J}(R)$. For all $b \in J$, we have $aRb \subseteq IJ \subseteq P$; hence, by (3), we have $b\langle s\rangle \subseteq P$, and thus, $J\langle s\rangle \subseteq P$. That completes the proof. □

From the above proposition, one can easily show that if R is a commutative ring with identity, and S is a multiplicatively closed subset of R, then Definitions 1 and 2 are equivalent.

Proposition 12. *Let R be a commutative ring with identity, and let S be a multiplicatively closed subset of R. Then, Definitions 1 and 2 are equivalent.*

Proof 27. Suppose P is an ideal of R that satisfies Definition 2, and let $ab \in P$ for some $a, b \in R$, then $\langle a\rangle\langle b\rangle \subseteq P$. Since S is a multiplicatively closed subset of R, then S is an m-system of R. Thus, by Definition 2, either $\langle a\rangle\langle s\rangle \subseteq P$ or $\langle b\rangle\langle s\rangle \subseteq P$ for some $s \in S$, and hence, either $as \in P$ or $bs \in P$. Now, suppose P satisfies Definition 1, and let $IJ \subseteq P$ for some ideals I, J of R. If $I\langle s\rangle \subseteq P$ and $J\langle s\rangle \subseteq P$ for all $s \in S$, then there exist $i \in I$ and $j \in J$ such that $i\langle s\rangle \subseteq P$ and $j\langle s\rangle \subseteq P$. However, $ij \in P$, thus by Definition 1, either $is \in P$ or $js \in P$ which implies either $i\langle s\rangle \subseteq P$ or $j\langle s\rangle \subseteq P$, a contradiction. □

Corollary 9. *Let P be an ideal of R, with $P \cap S = \varnothing$. If P is a right S-prime ideal such that $P \subseteq \mathcal{J}(R)$, then P is a right S-$\mathcal{J}$-ideal.*

Proof 28. Let $aRb \subseteq P$ and assume $a\langle s\rangle \nsubseteq \mathcal{J}(R)$ for $a, b \in R$. Then, $a\langle s\rangle \nsubseteq P$. Since P is right S-prime, then $b\langle s\rangle \subseteq P$. Thus by (3) of Proposition 11, P is right S-$\mathcal{J}$. □

The condition $P \subseteq \mathcal{J}(R)$ in Corollary 9 cannot be omitted in general. In the following, we give example to show this. But first, it is good to know that $\mathcal{J}(R[X]) = N[x]$, where $N = \mathcal{J}(R[x]) \cap R$, see Lemma 2J of [22]. In addition, the prime radical of $R[x]$, $\beta(R[x]) = \beta(R)[x]$, from Theorem 3 of [22]. Moreover, in Artinian rings, $\beta(R) = \mathcal{J}(R)$.

Example 5. Let $R = M_n(\mathbb{Z}_{36}[x])$. Since $\mathcal{J}(M_n(R)) = M_n(\mathcal{J}(R))$, (see [18] page 57) and R is Artinian, then

$$\begin{aligned}\mathcal{J}(R) &= \mathcal{J}(M_n(\mathbb{Z}_{36}[x])) = M_n(\mathcal{J}((\mathbb{Z}_{36}[x])) \\ &= M_n(\mathcal{J}(\mathbb{Z}_{36})[x]) = M_n(\langle 6\rangle[x])).\end{aligned} \tag{4}$$

Consider the m-system: $S = \{s, s^2, s^4, s^8, \ldots\}$, where $s = 9(e_{11} + e_{22} + \cdots + e_{nn})$, and $I = M_n(\langle\underline{9x}\rangle)$. It is clear that $S \cap I = \varnothing$ and $I \subseteq \mathcal{J}(R)$. Now, let $\mathcal{N}_1 R \mathcal{N}_2 \subseteq I$ for some matrices $\mathcal{N}_1, \mathcal{N}_2$ in R. Then, there exist some matrices $\mathcal{I}_1, \mathcal{I}_2$, such that either $\mathcal{N}_1 = x\mathcal{I}_1$ or $\mathcal{N}_2 = x\mathcal{I}_2$. Hence, either $\mathcal{N}_1\langle s\rangle \subseteq I$ or $\mathcal{N}_2\langle s\rangle \subseteq I$. Thus, I is right S-prime by Proposition 2.7 of [15]. However, by taking the ideals $A_1 = M_n(\langle 3x\rangle)$ and $A_2 = M_n(\langle 3\rangle)$, we have $A_1A_2 \subseteq I$; however, $A_1\langle s'\rangle \subseteq \mathcal{J}(R)$ and $A_2\langle s'\rangle \subseteq I$ for each $s' \in S$. Hence, I is not a right S-$\mathcal{J}$-ideal.

The following example presents an ideal which is right S-$\mathcal{J}$, but not a $\mathcal{J}$-ideal.

Example 6. Let $R = M_2(\mathbb{Z}_{12})$. Let

$$\begin{aligned}S_R &= \left\{\begin{bmatrix} s & 0 \\ 0 & s\end{bmatrix}; s \in S = \{1, 3, 9\}\right\}, \\ s' &= \begin{bmatrix} 3 & 0 \\ 0 & 3\end{bmatrix}.\end{aligned} \tag{5}$$

Then, S_R is an m-system of the ring R. Consider the ideal $P = M_2(\langle 4\rangle)$. Then, $\mathcal{J}(R) = M_n(\langle 6\rangle)$ and $P \cap S_R = \varnothing$. Since

$$\begin{aligned}&\begin{bmatrix} 2 & 0 \\ 0 & 2\end{bmatrix} R \begin{bmatrix} 2 & 0 \\ 0 & 2\end{bmatrix} \subseteq P, \\ &\begin{bmatrix} 2 & 0 \\ 0 & 2\end{bmatrix} \notin \mathcal{J}(R), \\ &\begin{bmatrix} 2 & 0 \\ 0 & 2\end{bmatrix} \notin P,\end{aligned} \tag{6}$$

then, P is not $\mathcal{J}$-ideal. Now, assume $\mathcal{N}_1 R \mathcal{N}_2 \subseteq P$, for some matrices $\mathcal{N}_1, \mathcal{N}_2 \in R$. If $\mathcal{N}_2\langle s'\rangle \subseteq P$, then P is right S_R-$\mathcal{J}$. Thus, assume $\mathcal{N}_2\langle s'\rangle \subseteq P$. Then, $\mathcal{N}_1 R \mathcal{N}_2 \subseteq M_2(\langle 2\rangle) = J$, and since J is prime, then either $\mathcal{N}_1 \in J$ or $\mathcal{N}_2 \in J$, now we discuss two cases as the following.

If $\mathcal{N}_1 \in J$, then $\mathcal{N}_1\langle s'\rangle \subseteq \mathcal{J}(R)$, and again P is right S_R-$\mathcal{J}$.

If $\mathcal{N}_1 \notin J$, then at least one of the entries of $\mathcal{N}_1$ is an odd number and $\mathcal{N}_2 \in J$; hence,

$$\mathcal{N}_2 = \begin{bmatrix} 2k_1 & 2k_2 \\ 2k_3 & 2k_4\end{bmatrix}, \quad \text{where } k_i \in \mathbb{Z}^+ \text{ for } i \in \{1, 2, 3, 4\}. \tag{7}$$

But by the initial assumption $\mathcal{N}_2\langle s'\rangle \subseteq P$, thus there exists

$$\mathcal{I} = \begin{bmatrix} 3\alpha_1 & 3\alpha_2 \\ 3\alpha_3 & 3\alpha_4\end{bmatrix} \in \langle s'\rangle, \tag{8}$$

where $\alpha_i \in \mathbb{Z}^+$, for $i \in \{1, 2, 3, 4\}$, such that $\mathcal{N}_2\mathcal{I} \notin P = M_2(\langle 4\rangle)$, which means that at least one of the entries of $\mathcal{N}_2\mathcal{I}$ is not a multiple of 4 in $\mathbb{Z}_{12}$. Without loss of generality, suppose that $6(k_1\alpha_1 + k_2\alpha_3)$ is not a multiple of 4, then either k_1 or/and k_2 is/are odd. Thus, at least one of k_i must be odd for $i \in \{1, 2, 3, 4\}$; consequently, at least one of the even entries of $\mathcal{N}_2$ is not a multiple of 4. By considering all the cases of the positions of the odd entries of $\mathcal{N}_1$ and the nonmultiple of 4 entries of $\mathcal{N}_2$, we can find, in each case, an element $\mathcal{M}$ of R, such that $\mathcal{N}_1\mathcal{M}\mathcal{N}_2 \notin P$, i.e., $\mathcal{N}_1 R \mathcal{N}_2 \subseteq P$, which is a contradiction. That is why $\mathcal{N}_1$ must be an element of J, and thus, $\mathcal{N}_1\langle s'\rangle \subseteq \mathcal{J}(R)$.

Theorem 7. *If R has an identity, and $P \lhd R$ with $P \cap S = \varnothing$, then, for some $s \in S$, $(P\colon \langle s\rangle)$ is a right S-$\mathcal{J}$-ideal, if and only if, P is right S-$\mathcal{J}$.*

Proof 29. Suppose $(P\colon \langle s\rangle)$ is a right S-$\mathcal{J}$-ideal, and let $xRy \subseteq P$ for some $x, y \in R$, then $xRy \subseteq (P\colon \langle s\rangle)$, hence by assumption, either $x\langle s_1\rangle \subseteq \mathcal{J}(R)$ or $y\langle s_1\rangle \subseteq (P\colon \langle s\rangle)$ for some $s_1 \in S$. If $y\langle s_1\rangle \subseteq (P\colon \langle s\rangle)$, then $y\langle s_1\rangle\langle s\rangle \subseteq P$, hence there exists $s' = s_1rs$ for some $r \in R$ such that $y\langle s'\rangle \subseteq y\langle s_1\rangle\langle s\rangle \subseteq P$. If $x\langle s_1\rangle \subseteq \mathcal{J}(R)$, then similarly, we find $x\langle s'\rangle \subseteq \mathcal{J}(R)$. Thus, P is right S-$\mathcal{J}$. Conversely, suppose P is right S-$\mathcal{J}$, and let $xRy \subseteq (P\colon \langle s\rangle)$ for some $x, y \in R$, then $\langle x\rangle\langle y\rangle\langle s\rangle \subseteq P$; hence, by assumption, either $\langle x\rangle\langle s_1\rangle \subseteq \mathcal{J}(R)$ or $\langle y\rangle\langle s\rangle\langle s_1\rangle \subseteq P$ for some $s_1 \in S$. If $\langle y\rangle\langle s\rangle\langle s_1\rangle \subseteq P$, then $\langle y\rangle\langle s\rangle\langle s_1\rangle \subseteq (P\colon \langle s\rangle)$; hence, there exists $s' = srs_1$ for some $r \in R$ such that $y\langle s'\rangle \subseteq \langle y\rangle\langle s\rangle\langle s_1\rangle \subseteq (P\colon \langle s\rangle)$. If $\langle x\rangle\langle s_1\rangle \subseteq \mathcal{J}(R)$, then similarly, we find $x\langle s'\rangle \subseteq \mathcal{J}(R)$. Thus, $(P\colon \langle s\rangle)$ is right S-$\mathcal{J}$. □

Proposition 13. *Suppose R has an identity, and $P \lhd R$ such that $P \cap S = \varnothing$. If for some $s \in S$, $(P\colon \langle s\rangle)$ is a $\mathcal{J}$-ideal, then P is right S-$\mathcal{J}$.*

Proof 30. Suppose $(P\colon \langle s\rangle)$ is a $\mathcal{J}$-ideal, and let $xRy \subseteq P$ for some $x, y \in R$, then $xRy \subseteq (P\colon \langle s\rangle)$, hence by assumption, either $x \in \mathcal{J}(R)$ or $y \in (P\colon \langle s\rangle)$. If $x \in \mathcal{J}(R)$, then $x\langle s\rangle \subseteq \mathcal{J}(R)$. If $y \in (P\colon \langle s\rangle)$, then $y\langle s\rangle \subseteq P$. Thus, by (3) of Proposition 11, P is right S-$\mathcal{J}$. □

The following proposition shows the converse of the above proposition holds when S is contained in the center of the ring $(C(R))$, and $(\mathcal{J}(R): \langle s\rangle)$ is a $\mathcal{J}$-ideal for some $s \in S$.

Proposition 14. *Let P be an ideal of a ring R with identity, S be an m-system of R such that $S \subseteq C(R)$, and $(\mathcal{J}(R): \langle s\rangle)$ be a $\mathcal{J}$-ideal, disjoint from S, for some $s \in S$. If P is right S-$\mathcal{J}$, then $(P: \langle s\rangle)$ is a $\mathcal{J}$-ideal.*

Proof 31. Since $(\mathcal{J}(R): \langle s\rangle)$ is a $\mathcal{J}$-ideal, then by Proposition 1.5 of [15], $(\mathcal{J}(R): \langle s\rangle) \subseteq \mathcal{J}(R)$, and hence, $(\mathcal{J}(R): \langle s\rangle) = \mathcal{J}(R)$. Let $xRy \subseteq (P: \langle s\rangle)$ for some $x, y \in R$, then $\langle x\rangle\langle y\rangle\langle s\rangle \subseteq P$, and by assumption, either $\langle x\rangle\langle s\rangle \subseteq \mathcal{J}(R)$ or $\langle y\rangle\langle s\rangle\langle s\rangle \subseteq P$. If $\langle x\rangle\langle s\rangle \subseteq \mathcal{J}(R)$, then $x \in (\mathcal{J}(R): \langle s\rangle) = \mathcal{J}(R)$. If $\langle y\rangle\langle s\rangle\langle s\rangle \subseteq P$, then $\langle s\rangle\langle s\rangle\langle y\rangle \subseteq P$, and again either $\langle s\rangle^3 \subseteq \mathcal{J}(R)$, which implies for some $r \in R$, $s' = srs \in \langle s\rangle^2 \subseteq (\mathcal{J}(R): \langle s\rangle) = \mathcal{J}(R)$, a contradiction, or $\langle y\rangle\langle s\rangle \subseteq P$, which implies $y \in (P: \langle s\rangle)$. Thus, $(P: \langle s\rangle)$ is a $\mathcal{J}$-ideal. □

Theorem 8. *Let $f: R_1 \longrightarrow R_2$ be a ring epimorphism, and P be a right S-$\mathcal{J}$-ideal of R_1 such that* $\operatorname{Ker}(f) \subseteq P$. *Then, $f(P)$ is a right $f(S)$-$\mathcal{J}$-ideal of R_2.*

Proof 32. Let $xR_2y \subseteq f(P)$ and $x\langle f(s)\rangle \subseteq \mathcal{J}(R_2)$ for $x, y \in R_2$. Since f is an epimorphism, there exists $c, d \in R_1$ such that $f(c) = x$ and $f(d) = y$. Thus, $xR_2y = f(c)f(R_1)f(d) = f(cR_1d) \subseteq f(P)$. Since $\operatorname{Ker}(f) \subseteq P$, we obtain $cR_1d \subseteq P$. Since P is S-$\mathcal{J}$ of R_1, then either $c\langle s\rangle \subseteq \mathcal{J}(R_1)$ or $d\langle s\rangle \subseteq P$ for some $s \in S$. If $c\langle s\rangle \subseteq \mathcal{J}(R_1)$, then, since $\mathcal{J}$ is a complete, idempotent Hoehnke radical, we have $f(\mathcal{J}(R_1)) \subseteq \mathcal{J}(R_2)$. Hence, $x\langle f(s)\rangle = f(c)\langle f(s)\rangle \subseteq f(\mathcal{J}(R_1)) \subseteq \mathcal{J}(R_2)$, that is, a contradiction. Hence, $c\langle s\rangle \subseteq \mathcal{J}(R_1)$, and thus, $d\langle s\rangle \subseteq P$. Since $y = f(d)$, then $y\langle f(s)\rangle \subseteq f(P)$; consequently, $f(P)$ is a right $f(S)$-$\mathcal{J}$-ideal of R_2. □

Theorem 9. *Let $f: R_1 \longrightarrow R_2$ be a ring epimorphism, and P be an ideal of R such that* $\operatorname{Ker}(f) \subseteq P \cap \mathcal{J}(R)$ *and $P \cap S = \varnothing$. If $f(P)$ is a right $f(S)$-$\mathcal{J}$ of R_2, then P is a right S-$\mathcal{J}$ of R_1.*

Proof 33. Suppose that $A_1B_1 \subseteq P$ for ideals A_1, B_1 of R. Then, $f(A_1)f(B_1) = f(A_1B_1) \subseteq f(P)$. Since $f(P)$ is a right $f(S)$-$\mathcal{J}$ of R_2, then for some $s \in S$,

$$\text{either } f(B_1)\langle f(s)\rangle \subseteq f(P), \text{or } f(A_1)\langle f(s)\rangle \subseteq \mathcal{J}(R_2). \tag{9}$$

□

If $f(B_1)\langle f(s)\rangle \subseteq f(P)$, then $B_1\langle s\rangle \subseteq f^{-1}(f(B_1\langle s\rangle)) \subseteq f^{-1}(f(P)) = P$, since $\operatorname{Ker}(f) \subseteq P$. If $f(A_1)\langle f(s)\rangle \subseteq \mathcal{J}(R_2)$, in this case, we show that $A_1\langle s\rangle \subseteq \mathcal{J}(R_1)$ as the following. Let J be an ideal of R_1 with $R_1/J \in \mathcal{S}_{\mathcal{J}} \cap \mathcal{K}$, then $\mathcal{J}(R_1) \subseteq J$, thus $\operatorname{Ker}(f) \subseteq J$. Hence, $f(R_1)/f(J) \cong (R_1/\operatorname{Ker}(f))/(J/\operatorname{Ker}(f)) \cong R_1/J$, so $R_2/f(J) = f(R_1)/f(J) \in \mathcal{S}_{\mathcal{J}} \cap \mathcal{K}$, and thus $f(A_1)\langle f(s)\rangle \subseteq \mathcal{J}(R_2) \subseteq f(J)$; thus $A_1\langle s\rangle \subseteq f^{-1}(f(A_1\langle s\rangle)) \subseteq f^{-1}(f(J)) = J$ for each J in $\cap\{I \triangleleft R: R/I \in \mathcal{S}_{\mathcal{J}} \cap \mathcal{K}\}$, and hence, $A_1\langle s\rangle \subseteq \mathcal{J}(R_1)$. Thus, P is the right S-$\mathcal{J}$ of R_1.

Proposition 15. *Let R be a ring with identity, $P \triangleleft R$ and $(\mathcal{J}(R): \langle s\rangle) = \mathcal{J}(R)$ for some $s \in S$. The following are equivalent:*

1. *P is a right S-$\mathcal{J}$-ideal of R associated with $s \in S$.*
2. *(i) $P \subseteq (\mathcal{J}(R): \langle s\rangle)$. (ii) For all $x, y \in R$ with $xRy \subseteq P$, either $x \in (\mathcal{J}^*(P): \langle s\rangle)$ or $y \in (P: \langle s\rangle)$.*

Proof 34. (1) ⇒ (2) (i) Suppose $P \subseteq (\mathcal{J}(R): \langle s\rangle)$, then for all $x \in P \setminus (\mathcal{J}(R): \langle s\rangle)$ and any $y \in R$, we have $\langle x\rangle\langle y\rangle \subseteq P$; hence, by (1), we obtain $y \in (P: \langle s\rangle)$ for some $s \in S$, and thus, $R = (P: \langle s\rangle)$, which implies $1\langle s\rangle \subseteq P$, a contradiction, since $P \cap S = \varnothing$. Thus, $P \subseteq (\mathcal{J}(R): \langle s\rangle)$.

(ii) Now, assume $xRy \subseteq P$ for some x, y of R. By (3) of Proposition 11, either $x \in (\mathcal{J}(R): \langle s\rangle) \subseteq (\mathcal{J}^*(P): \langle s\rangle)$ or $y \in (P: \langle s\rangle)$ for some $s \in S$.

(2) ⇒ (1) Assume $A_1A_2 \subseteq P$ for some ideals A_1, A_2 of R. Assume that $A_1\langle s\rangle \subseteq \mathcal{J}(R)$ and $A_2\langle s\rangle \subseteq P$ for all $s \in S$. Then, there is $a_1 \in A_1 \setminus (\mathcal{J}(R): \langle s\rangle)$ and $a_2 \in A_2 \setminus (P: \langle s\rangle)$. Hence, $a_1Ra_2 \subseteq A_1A_2 \subseteq P$, and by (ii), we obtain either $a_1 \in (\mathcal{J}^*(P): \langle s\rangle)$ or $a_2 \in (P: \langle s\rangle)$. Now, since $P \subseteq (\mathcal{J}(R): \langle s\rangle) = \mathcal{J}(R)$, then $\mathcal{J}^*(P) = \mathcal{J}(R)$. Thus, either $a_1 \in (\mathcal{J}(R): \langle s\rangle)$ or $a_2 \in (P: \langle s\rangle)$, a contradiction in both cases. Hence, P is right S-$\mathcal{J}$ of R. □

Proposition 16. *Let R be an identity and $P \triangleleft R$ such that $S \cap P = \varnothing$.*

1. *If P is a right S-$\mathcal{J}$-ideal, then $P \subseteq (\mathcal{J}(R): \langle s\rangle)$ for some $s \in S$*
2. *$\mathcal{J}(R)$ is a right S-$\mathcal{J}$, if and only if, $\mathcal{J}(R)$ is a right S-prime*

Proof 35

1. Assume P is a right S-$\mathcal{J}$-ideal. For all $a \in P$, $\langle a\rangle\langle s_1\rangle \subseteq P$, for some $s_1 \in S$. Hence, either $\langle a\rangle\langle s\rangle \subseteq \mathcal{J}(R)$ or $\langle s_1\rangle\langle s\rangle \subseteq P$ for some $s \in S$. If $\langle s_1\rangle\langle s\rangle \subseteq P$, then for some $r \in R$, $s_2 = s_1rs \in S$; hence, $\langle s_2\rangle \subseteq \langle s_1\rangle\langle s\rangle \subseteq P$, a contradiction, because $S \cap P = \varnothing$. Hence, $\langle a\rangle\langle s\rangle \subseteq \mathcal{J}(R)$ for all $a \in P$, and thus, $P \subseteq (\mathcal{J}(R): \langle s\rangle)$.
2. The proof is routine. □

Recall that an ideal P of a ring R is called superfluous if $A + B = R$ for some ideal B of R, then $B = R$.

Corollary 10. *For a local ring R with a unique maximal ideal $\mathcal{M}$, if $(\mathcal{J}(R): \langle s\rangle)$ is a $\mathcal{J}$-ideal of R, then I is a superfluous ideal for any right S-$\mathcal{J}$-ideal I of R.*

Proof 36. Suppose $I + J = R$ for some ideal J of R, then there exist $i \in I$ and $j \in J$ such that $i + j = 1$; hence, $1 - j = i \in I$. Now, since I is right S-$\mathcal{J}$ of R, then by Proposition 15, $I \subseteq (\mathcal{J}(R): \langle s\rangle)$, and since $(\mathcal{J}(R): \langle s\rangle)$ is a $\mathcal{J}$-ideal, then by Proposition 1.5 of [15], $(\mathcal{J}(R): \langle s\rangle) \subseteq \mathcal{J}(R) \subseteq (\mathcal{J}(R): \langle s\rangle)$. Consequently,

$$1 - j = i \in I \subseteq (\mathcal{J}(R) \colon \langle s \rangle) = \mathcal{J}(R) = \mathcal{M}. \tag{10}$$

□

Thus, j is a unit, and hence, $J = R$, and I is superfluous.

It is interesting to explore the structure of rings where every ideal is an S-$\mathcal{J}$-ideal. As noted in [25], Proposition 4.5 demonstrates that in the noncommutative Artinian local rings, every ideal is a $\mathcal{J}$-ideal and therefore also an S-$\mathcal{J}$-ideal. However, the question remains open regarding the structure of rings where every ideal is an S-$\mathcal{J}$-ideal but not necessarily a $\mathcal{J}$-ideal, that is, where there exists at least one ideal that is not a $\mathcal{J}$-ideal.

3.1. Conclusion. In this paper, we have introduced and explored the concept of S-$\mathcal{J}$-ideals in both commutative and noncommutative rings. We demonstrated that many properties of $\mathcal{J}$-ideals carry over to S-$\mathcal{J}$-ideals, thus offering a strong generalization. Our investigation into various ring constructions, including homomorphic image rings, quotient rings, cartesian product rings, polynomial rings, power series rings, idealization rings, and amalgamation rings, reveals the versatility and broad applicability of S-$\mathcal{J}$-ideals.

In noncommutative rings, we have defined right S-$\mathcal{J}$-ideals where S is an m-system and demonstrated their equivalence with S-$\mathcal{J}$-ideals in the commutative case with identity. The provided examples illustrate the connections and distinctions between right S-prime ideals and $\mathcal{J}$-ideals, enriching the understanding of the new concept.

This new class of ideals paves the way for further studies, such as weakly S-$\mathcal{J}$ and quasi S-$\mathcal{J}$-ideals, in both commutative and noncommutative rings. These are closely related examples, and one can also, for example, explore S-$\mathcal{J}$-primary ideals in future research.

We conclude the paper with several open questions that arise naturally from our investigation:

1. Suppose the localization IS^{-1} is a $\mathcal{J}$-ideal of RS^{-1}. Does it follow that I is an S-$\mathcal{J}$-ideal of R?
2. Is the set S used in the localization the same as the one used in our generalization to define S-$\mathcal{J}$-ideals?
3. How can we characterize rings where every ideal (disjoint from S) is an S-$\mathcal{J}$-ideal but not necessarily a $\mathcal{J}$-ideal?

These questions remain unanswered but open up avenues for further research.

Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Disclosure

A preprint has previously been published [1].

Conflicts of Interest

The authors declare no conflicts of interest.

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