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# Orbital stability of periodic standing waves for the cubic fractional nonlinear Schrödinger equation

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**Abstract**

In this paper, the existence and orbital stability of the periodic standing wave solutions for the nonlinear fractional Schrödinger (fNLS) equation with cubic nonlinearity is studied. The existence is determined by using a minimizing constrained problem in the complex setting and it is showed that the corresponding real solution is always positive. The orbital stability is proved by combining some tools regarding the oscillation theorem for fractional Hill operators and the Vakhitov-Kolokolov condition, well known for Schrödinger equations. We then perform a numerical approach to generate the periodic standing wave solutions of the fNLS equation by using the Petviashvili's iteration method. We also investigate the Vakhitov-Kolokolov condition numerically which cannot be obtained analytically for some values of the order of the fractional derivative.


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*Keywords:* Fractional Schrödinger equation; Existence and uniqueness of minimizers; Small-amplitude periodic waves; Orbital stability

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## 1. Introduction

In this paper, we present results concerning the existence and orbital stability of periodic standing waves for the fractional nonlinear Schrödinger equation (fNLS) in the focusing case given as

$$iu_t - (-\Delta)^s u + |u|^2 u = 0. \tag{1.1}$$

Here, $u : \mathbb{T} \times \mathbb{R} \longrightarrow \mathbb{C}$ is a complex-valued function and $2\pi$-periodic with respect to the first variable with $\mathbb{T} := [-\pi, \pi]$. The fractional Laplacian $(-\Delta)^s$ is defined as a pseudo-differential operator

$$\widehat{(-\Delta)^s g}(\xi) = |\xi|^{2s}\widehat{g}(\xi), \tag{1.2}$$

where $\xi \in \mathbb{Z}$ and $s \in (0, 1]$ (see [54]). The fNLS equation was introduced by Laskin in [39] and [40] and it appears in several physical applications such as fluid dynamics, quantum mechanics, in the description of Boson stars and water wave dynamics ([35], [37] and [52]).

Equation (1.1) admits the conserved quantities $E, F : H^s_{per} \to \mathbb{R}$ which are given as

$$E(u) = \frac{1}{2}\int_{-\pi}^{\pi} |(-\Delta)^{\frac{s}{2}} u|^2 - \frac{1}{2}|u|^4 \, dx, \tag{1.3}$$

and

$$F(u) = \frac{1}{2}\int_{-\pi}^{\pi} |u|^2 \, dx. \tag{1.4}$$

When $s = 1$, we obtain that $(-\Delta)^s = -\Delta$ is the well known Laplacian operator and (1.1) reduces to the cubic nonlinear Schrödinger equation (NLS) in the focusing case. As far as we know, there exist many applications for this specific equation such as optics, quantum mechanics, Bose-Einstein condensates, laser beam propagation and DNA modelling. In mathematical point of view, the NLS equation describes nonlinear waves and dispersive wave phenomena ([9], [12], [23] and [56]). In addition, there are many qualitative aspects concerning this equation and one of them is the orbital stability of standing/travelling solitary waves in one or higher dimensions. We refer the reader to [13], [28], [29], [43], [55], and [60] for more detailed discussions.
A standing periodic wave solution for the equation (1.1) has the form

$$u(x, t) = e^{i\omega t}\varphi(x), \tag{1.5}$$

where $\varphi : \mathbb{T} \longrightarrow \mathbb{R}$ is a smooth $2\pi$-periodic function and $\omega \in \mathbb{R}$ represents the wave frequency which is assumed to be positive. Substituting (1.5) into (1.1), we obtain the following differential equation with fractional derivative

$$(-\Delta)^s\varphi + \omega\varphi - \varphi^3 = 0. \tag{1.6}$$

For $\omega > 0$, let us consider the standard Lyapunov functional defined as

$$G(u) := E(u) + \omega F(u). \tag{1.7}$$

By (1.6), we obtain $G'(\varphi, 0) = 0$, that is, $(\varphi, 0)$ is a critical point of $G$. In addition, the linearized operator around the pair $(\varphi, 0)$ is given by

$$\mathcal{L} := G''(\varphi, 0) = \begin{pmatrix} \mathcal{L}_1 & 0 \\ 0 & \mathcal{L}_2 \end{pmatrix}, \tag{1.8}$$

where

$$\mathcal{L}_1 = (-\Delta)^s + \omega - 3\varphi^2 \qquad \text{and} \qquad \mathcal{L}_2 = (-\Delta)^s + \omega - \varphi^2. \tag{1.9}$$

Both operators $\mathcal{L}_1$ and $\mathcal{L}_2$ are self-adjoint and they are defined in $L^2_{per}$ with dense domain $H^{2s}_{per}$. Operator $\mathcal{L}$ in (1.8) plays an important role in our study.

For the case $s = 1$, we have the pioneer work of Angulo [4] where the author established results of orbital stability for positive and periodic standing waves with dnoidal profile. For this aim, the author combined the classical Floquet theory for the Hill operators $\mathcal{L}_1$ and $\mathcal{L}_2$ in (1.9) with the stability approaches in [28] and [60]. In the interesting work of Gustafson *et al.* in [30], the authors obtained cnoidal periodic wave solutions using a variational method to prove spectral stability results with respect to perturbations with the same period $L$ and orbital stability results in the space constituted by anti-periodic functions with period $L/2$. Deconinck and Upsal in [19] used the integrability of the NLS equation to determine orbital stability results for the dnoidal waves with respect to subharmonic perturbations in the space of continuous bounded functions. Additional references concerning orbital/spectral stability of periodic waves can be found in [8], [14], [18], [25], [26], [27], [42] and [46].

When $s \in (0, 1)$, the orbital stability of real-valued, even and anti-periodic standing wave solutions $\psi$ of (1.1) has been studied by Claassen and Johnson in [16]. The authors determined the existence of real solutions via a minimization problem in the context of anti-periodic functions (denoted by $L^2_a(0, L)$) and they established that the associated linearized operator acting in $L^2_a(0, L)$ is non-degenerate. By assuming the additional assumption $\frac{d}{d\omega}\int_0^L \psi^2 dx > 0$ (the well-known Vakhitov-Kolokolov condition), the authors were able to show that $\psi$ is orbitally stable with respect to anti-periodic perturbations in a suitable subspace of $H^s(0, L) \cap L^2_a(0, L)$.

Hakkaev and Stefanov in [31] have determined the existence and the orbital (spectral) stability of positive and periodic single-lobe solutions $\phi$ for the quadratic fractional Schrödinger equation

$$iu_t - (-\Delta)^s u + |u|u = 0, \tag{1.10}$$

where $s \in \left(\frac{1}{4}, 1\right)$. For the existence of periodic minimizers and stability, the authors used a (real) minimization problem as

$$\inf\left\{\mathscr{E}(v) := \frac{1}{2}\int_{-1}^{1} ((-\Delta)^{\frac{s}{2}} v)^2\, dx - \frac{1}{3}\int_{-1}^{1} v^3\, dx\ ;\ v \in H^s_{per}([-1, 1]),\ \int_{-1}^{1} v^2\, dx = \lambda\right\}, \tag{1.11}$$

where $\lambda > 0$ is given. It is important to note that if a minimization problem as in (1.11) is solved, the spectral stability of periodic waves can be established. According to [28], [29], [47] and [60] it is necessary to determine that:

i) $\mathrm{n}(\mathcal{L}) = 1$ and $\mathrm{Ker}(\mathcal{L}) = [(\phi', 0), (0, \phi)]$, where $\mathrm{n}(\mathcal{L})$ stands for the number of negative eigenvalues of $\mathcal{L}$,

ii) $\frac{d}{d\omega}\int_{-1}^{1} \phi^2 dx > 0$,

for the orbital stability. The first condition has been proved by the authors using that the solution $\phi$ which solves the minimizing problem (1.10) is positive (since it satisfies the equation $\phi^2 = ((-\Delta)^s + \omega)\phi$, where $\omega > 0$) and an oscillation theorem which is determined in [16].

Our aim in this work is to show that the standing wave solution in (1.5), where $\varphi = \varphi_\omega$ is a positive and single-lobe periodic wave (see Definition 3.1), is orbitally stable/unstable. According to the sufficient conditions for the orbital stability in the energy space $H^s_{per}$ in [28], we need to analyse the local and global well-posedness of the associated Cauchy problem for the fNLS equation (1.1). For this important topic, we first refer to the study [7] by Boling, Yongqian and Jie. They have used Galerkin's method to give the global well-posedness results for the $n$-dimensional Cauchy problem

$$\begin{cases} iu_t + (-\Delta)^s u + \beta |u|^\rho u = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{1.12}$$

For $s > \frac{n}{2}$, global solutions in $H^s_{per}(\mathbb{T}^n)$ were established when $\beta > 0$ and $\rho > 0$. If $0 < s < \frac{n}{2}$, it is necessary to assume $\rho \in \left(0, \frac{4s}{n-2s}\right)$ to obtain the same result. For the case $\beta < 0$, the condition for the existence of global solutions is $\rho \in \left(0, \frac{4s}{n}\right)$. Demirbas, Erdoğan and Tzirakis in [20] have studied the existence and uniqueness for the Cauchy problem (1.12) for the case $n = 1$, $\rho = 2$ and $\beta = 1$. Using Gagliardo-Nirenberg inequality, tools of Bourgain spaces and Strichartz estimates, the authors determined the existence of local solutions in $H^\alpha_{per}(\mathbb{T})$ for $\alpha > \frac{1-s}{2}$ and global solutions for $\alpha > \frac{10s+1}{12}$. Cho, Hwang, Kwon and Lee in [15] used Bourgain spaces to establish local solutions in $H^\alpha_{per}(\mathbb{T})$ for $\alpha \geqslant \frac{1-s}{2}$. A refined result concerning the local well-posedness for the case $\beta = -1$ is given in [57].

A common misunderstanding made by some authors is to apply the Gagliardo-Nirenberg inequality in the periodic case by borrowing out the well-known result determined in the whole line. In fact, concerning the Cauchy problem associated with the equation (1.1), the authors in [7] have used the inequality

$$\|f\|^4_{L^4_{per}} \leqslant C \|(-\Delta)^{\frac{s}{2}} f\|^{\frac{1}{s}}_{L^2_{per}} \|f\|^{4-\frac{1}{s}}_{L^2_{per}}, \tag{1.13}$$

where $f \in H^s_{per}$ and $C > 0$ is a constant not depending on $f$. As far as we know, the inequality (1.13) is not true because it fails when $f \in H^s_{per}$ is a non-zero constant. In [20] the authors claim that global solutions of (1.1) are established by using the same inequality as in (1.13) in the periodic context. However, since they are considering the equation posed both on the torus and the real line, the authors disregard that (1.13) can not be used in this specific case.

To the best of our knowledge, an additional term containing the $L^2$-norm needs to be added to (1.13) since it is deduced from the well-known inequality posed in bounded domains (see [48]). It is important to note that the additional term containing the $L^2$-norm does not intervene in the analysis of existence of global solutions for the Cauchy problem associated with equation (1.1) since the $L^2$-norm is a conserved quantity. Besides the orbital stability/instability results, our intention is to present a precise statement concerning the Gagliardo-Nirenberg inequality in the periodic context given by

$$\|f\|^4_{L^4_{per}(\mathbb{T})} \leqslant C\|(-\Delta)^{\frac{s}{2}} f\|^{\frac{1}{s}}_{L^2_{per}(\mathbb{T})} \|f\|^{4-\frac{1}{s}}_{L^2_{per}(\mathbb{T})} + C\|f\|^4_{L^2_{per}(\mathbb{T})}. \tag{1.14}$$

We now give the main points of our paper: First, we show the existence of an even periodic single-lobe solution $\varphi$ for the equation (1.6). Let $\tau > 0$ be fixed. Following similar arguments as in [44] and [45], we need to solve the following constrained minimization problem

$$\inf\left\{\mathcal{B}_\omega(u) := \frac{1}{2}\int_{-\pi}^{\pi} |(-\Delta)^{\frac{s}{2}} u|^2 + \omega|u|^2\, dx\ ;\ u \in H^s_{per,e},\ \int_{-\pi}^{\pi} |u|^4\, dx = \tau\right\}, \tag{1.15}$$

where $\omega > 0$ and $s \in \left(\frac{1}{4}, 1\right]$. Different from the approaches [31], [44] and [45], we see that $u$ in (1.15) is complex, so that the eventual solution $\Phi$ for the mentioned problem is a complex-valued function. For every $\theta \in \mathbb{R}$, we see that $e^{-i\theta}\Phi$ is also a minimizer for (1.15), so that we can assume $\Phi = e^{i\theta_0}\varphi$, where $\theta_0 \in \mathbb{R}$ is a fixed real number and $\varphi$ is a real-valued $2\pi$-periodic function (see Remarks 3.3, 3.4 and 3.5 in Section 3 for additional details). This assumption enables to consider a real valued solution $\varphi$ for the problem (1.15) which is even. In addition, we can consider that $\varphi$ has a single-lobe profile for all $\omega > \frac{1}{2}$ (see Proposition 3.6).

Another way to construct periodic real valued solutions for the equation (1.6) can be determined by using the local and global bifurcation theory in [11]. First, we construct small amplitude periodic solutions in the same way as in [44] (see also [10]) for $\omega > \frac{1}{2}$ and close to the bifurcation point $\frac{1}{2}$. After that, we give sufficient conditions to extend parameter $\omega$ to the whole interval $(\frac{1}{2}, +\infty)$ by constructing an even periodic continuous function $\omega \in \left(\frac{1}{2}, +\infty\right) \longmapsto \varphi_\omega \in H^{2s}_{per,e}$ where $\varphi_\omega$ solves equation (1.6). However, since the periodic wave obtained by the global bifurcation theory may not have a single-lobe profile, we choose the periodic waves which arise as a minimum of the problem (1.15). The existence of small amplitude waves associated with the Schrödinger equation were determined in [25] for the equation (1.12) with $s = 1$ and $\beta = \pm 1$. First they show that these waves are orbitally stable within the class of solutions which have the same period. For the case of general bounded perturbations, they prove that the small amplitude travelling waves are stable in the defocussing case and unstable in the focusing case.

The fact that the minimizer $\varphi$ of (1.15) is a real even single-lobe solution for (1.6) gives us useful spectral properties which in turn play an important role regarding our stability approach. Using the fact that $\varphi$ minimizes the constrained problem in (1.15), we see that $\mathrm{n}(\mathcal{L}) = 1$. Since $\mathcal{L}$ in (1.8) is a diagonal operator, it is possible to obtain by the fact $(\mathcal{L}_1\varphi, \varphi)_{L^2_{per}} = -2\int_{-\pi}^{\pi} \varphi^4 dx < 0$ that $\mathrm{n}(\mathcal{L}_1) = 1$ and $\mathrm{n}(\mathcal{L}_2) = 0$ (see Section 2 for the precise notations of $\mathrm{n}(\mathcal{L}_i)$, $i = 1, 2$). This means by the fact $\mathcal{L}_2\varphi = 0$ that $0$ is the first eigenvalue for $\mathcal{L}_2$. A simple application of the standard Krein-Ruttman Theorem gives us $\varphi > 0$, so that the solution is positive. Next, if the periodic minimizer $\varphi$ obtained in the minimization problem in (1.15) depends smoothly on $\omega \in \left(\frac{1}{2}, +\infty\right)$, we obtain by the fact $\varphi$ is a positive even single-lobe that $\mathrm{z}(\mathcal{L}_1) = 1$, that is, $\mathrm{Ker}(\mathcal{L}_1) = [\varphi']$. Here, the positivity of the single-lobe profile plays an important role in our spectral analysis since it avoids the additional assumption $1 \in \mathrm{R}(\mathcal{L}_1)$ as required in [33], [44] and [45] to obtain that $\mathrm{z}(\mathcal{L}_1) = 1$. All facts concerning the spectral analysis for the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ in (1.9) enable us to conclude, since $\mathcal{L}$ in (1.8) is a diagonal operator, that $\mathrm{n}(\mathcal{L}) = 1$ and $\mathrm{z}(\mathcal{L}) = 2$.

The strategy to prove the orbital stability is based on an adaptation of the arguments in [28] and [47] to the periodic setting. Notice that $\mathrm{n}(\mathcal{L}) = 1$ and $\mathrm{z}(\mathcal{L}) = 2$ are useful to consider the standing wave solution in (1.5) containing only one symmetry (rotation), but the orbital stability can be considered with the orbit generated by the wave $\varphi$ containing two symmetries (namely, rotation and translation). To do so, we need to employ the stability result in [47] and the existence of global solutions in time is a cornerstone in our analysis. Since we can obtain a global well-posedness result for the case $s \in \left(\frac{1}{2}, 1\right]$ according to the inequality (1.14), the orbital stability of the wave can be established provided that $\mathsf{q} := \frac{d}{d\omega}\int_{-\pi}^{\pi} \varphi^2 dx > 0$. The stability result in [28] can be also used for the orbital stability and yields $\mathsf{q} > 0$. However, we need to consider only one basic symmetry for the orbit and since we consider standing waves of the form (1.5), it is natural to consider the orbit generated by the wave constituted only by rotations. In the latter case, the energy space is the periodic Sobolev space $H^s_{per,e}$ restricted to the even functions instead of the usual energy space $H^s_{per}$. Restricted to this new space $L^2_{per,e}$, we have $\mathrm{n}(\mathcal{L}) = \mathrm{z}(\mathcal{L}) = 1$ and this fact agrees well with the spectral (sufficient) conditions for the orbital stability in [28].

Concerning the orbital instability, we can apply the *instability theorem* in [28] and the fact that $\mathrm{n}(\mathcal{L}) = \mathrm{z}(\mathcal{L}) = 1$ over the space $L^2_{per,e}$. Note that the orbital instability in the space $H^s_{per,e}$ will be considered in the orbit generated again by a single symmetry. Even though we are considering a smaller subspace, the orbital instability can be considered in the whole energy space $H^s_{per}$ and the orbit generated by the two symmetries. To do so, the only requirement is that $\mathsf{q} < 0$. The above results yield the theorem:

**Theorem 1.1.** *Let $\varphi = \varphi_\omega$ be the positive and periodic single-lobe solution for the equation* (1.6) *obtained in Proposition* 3.6*, for all $\omega \in (\frac{1}{2}, +\infty)$. If $\varphi$ depends smoothly on $\omega \in \left(\frac{1}{2}, +\infty\right)$, the periodic wave is orbitally stable if $\mathsf{q} > 0$. If $\mathsf{q} < 0$, the periodic wave is orbitally unstable.*

To obtain the sign of the quantity $\mathsf{q}$ we use a numerical approach. For this aim, we first use the Petviashvili's iteration method to generate the periodic standing wave solutions of the fNLS equation. Then we check whether $\int_{-\pi}^{\pi} \varphi^2 dx$ is increasing or decreasing with respect to $\omega$ to determine the sign of $\mathsf{q}$. Our results are then established:

**Theorem 1.2.** *Suppose that assumptions in Theorem* 1.1 *are satisfied.*
*i) If $s \in \left(\frac{1}{4}, \frac{1}{2}\right]$ the periodic wave $\varphi$ is orbitally unstable.*
*ii) There exists $s^* \approx 0.6$ such that if $s \in \left[s^*, 1\right]$ the periodic wave $\varphi$ is orbitally stable.*
*iii) For $s \in \left(\frac{1}{2}, s^*\right)$, there exists a critical value $\omega_c > \frac{1}{2}$ such that the periodic wave $\varphi$ is orbitally unstable if $\omega \in \left(\frac{1}{2}, \omega_c\right)$ and orbitally stable if $\omega \in (\omega_c, +\infty)$.*

**Remark 1.3.** It is important to mention that the sense of orbital stability mentioned in Theorem 1.2 (see Definition 5.1 for further details) prescribes the existence of global solutions in the energy space $H^s_{per}$. As far as we can see, if $s \in \left(\frac{1}{2}, 1\right]$ Proposition 2.5 in the next section establishes global well-posedness in $H^s_{per}$ for initial data in the same space (consequently, the orbital stability/instability according to the Theorem 1.2) and for $s \in \left(\frac{1}{4}, \frac{1}{2}\right]$, we do not know a suitable result of local well-posedness. To overcome this difficulty, we need to consider a smooth initial data $u_0$ in $H^\alpha_{per}$ for $\alpha > s$ large enough in order to obtain at least the existence of smooth solutions. Since the periodic standing wave $\varphi$ is also smooth we obtain, in fact, a conditional (in)stability result for smooth solutions.

Our paper is organized as follows: In Section 2, we show the Gagliardo-Nirenberg inequality for fractional operators in the periodic context. The existence of even periodic minimizers with a single-lobe profile as well as the existence of small amplitude periodic waves are determined in Section 3. In Section 4, we present spectral properties for the linearized operator related to the fNLS equation and some results concerning the uniqueness of minimizers. Finally, our result about orbital stability and instability associated with periodic waves is shown in Section 5.

**Notation.** For $s \geqslant 0$, the real/complex Sobolev space $H^s_{per} := H^s_{per}(\mathbb{T})$ consists of all periodic distributions $f$ such that

$$\|f\|^2_{H^s_{per}} := 2\pi \sum_{k=-\infty}^{\infty} (1+k^2)^s |\hat{f}(k)|^2 < \infty, \tag{1.16}$$

where $\hat{f}$ is the periodic Fourier transform of $f$ and $\mathbb{T} = [-\pi, \pi]$. The space $H^s_{per}$ is a Hilbert space with the natural inner product denoted by $(\cdot,\cdot)_{H^s_{per}}$. When $s = 0$, the space $H^s_{per}$ is isometrically isomorphic to the space $L^2_{per} := H^0_{per}$ (see, e.g., [36]). The norm and inner product in $L^2_{per}$ will be denoted by $\|\cdot\|_{L^2_{per}}$ and $(\cdot,\cdot)_{L^2_{per}}$, respectively. We omit the interval $[-\pi, \pi]$ of the space $H^s_{per}(\mathbb{T})$ and we denote it by $H^s_{per}$ shortly. In addition, the norm in (1.16) can be written as (see [3])

$$\|f\|^2_{H^s_{per}} = \|(-\Delta)^{\frac{s}{2}} f\|^2_{L^2_{per}} + \|f\|^2_{L^2_{per}}. \tag{1.17}$$

For $s \geqslant 0$, the space $H^s_{per,e} := \{f \in H^s_{per} \; ; \; f \text{ is an even function}\}$ is endowed with the same norm and inner product in $H^s_{per}$. If it is needed, the above notations can be extended in the complex/vectorial case in the following sense: $f \in H^s_{per} \times H^s_{per}$ we have $f = f_1 + i f_2 \equiv (f_1, f_2)$, where $f_i \in H^s_{per}$ $(i = 1, 2)$ since $\mathbb{C}$ is identified with $\mathbb{R}^2$.

We denote the number of negative eigenvalues and the dimension of the kernel of a certain linear operator $\mathcal{A}$, by $\mathrm{n}(\mathcal{A})$ and $\mathrm{z}(\mathcal{A})$, respectively.

## 2. Gagliardo-Nirenberg inequality in the fractional periodic context

In this section, we show the Gagliardo-Nirenberg inequality for fractional operators in the periodic case. Our intention is to give a precise result of global well-posedness associated with the following Cauchy problem

$$\begin{cases} iu_t - (-\Delta)^s u + u|u|^2 = 0, \\ u(x, 0) = u_0(x). \end{cases} \tag{2.1}$$

For this aim, we need the Gagliardo-Nirenberg inequality for bounded domains of cone-type $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$ (for details of this kind of domains, see [58, Section 4.2.3, Equation 7]) stated in the next lemma. In the rest of this section, we consider the fractional Sobolev space $H^r_q(\Omega) = W^{r,q}(\Omega)$, well known as *Slobodeckij space* (for details, see [6, Section 1.2], [58, Section 2.3.3, Equation 1] and [58, Section 4.2.1, Definition 1]) for each $r \in [0, 1)$ and $q \geqslant 1$. In what follows, we handle with real-valued functions. For complex-valued functions, the arguments are similar.

**Lemma 2.1** *(Gagliardo-Nirenberg inequality for bounded domains of cone-type). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of cone-type. If $k, s \in (0, 1)$, $p, p_0 > 1$ and $r > 0$ satisfy*

$$r = ks \quad \textit{and} \quad \frac{1}{p} = \frac{1-k}{p_0} + \frac{k}{2},$$

*then there exists $C_1 > 0$ such that,*

$$\|f\|_{H^r_p(\Omega)} \leqslant C_1 \|f\|^{1-k}_{L^{p_0}(\Omega)} \|f\|^k_{H^s(\Omega)}, \tag{2.2}$$

*for all $f \in L^{p_0}(\Omega) \cap H^s(\Omega)$.*

**Proof.** First of all, according to [58, Section 4.3.1, Theorem 2] the relation of interpolation

$$(H^{s_0}_{p_0}(\Omega), H^s_{p_1}(\Omega))_k = H^r_p(\Omega),$$

is valid. Here, $p, p_1, p_0 > 1$, $k \in (0, 1)$, $s_0, s \geqslant 0$, and $r > 0$ satisfy

$$r = s_0(1-k) + ks \quad \text{and} \quad \frac{1}{p} = \frac{1-k}{p_0} + \frac{k}{p_1}.$$

As a consequence of [58, Section 1.3.3, Equation 5] there exists a constant a $C_0 > 0$ such that

$$\|f\|_{H^r_p(\Omega)} \leqslant C_0 \|f\|^{1-k}_{H^{s_0}_{p_0}(\Omega)} \|f\|^k_{H^s_{p_1}(\Omega)}, \tag{2.3}$$

for all $f \in H^{s_0}_{p_0}(\Omega) \cap H^s_{p_1}(\Omega)$. In particular, by considering $p_1 = 2$, $s_0 = 0$ and $s \in (0, 1)$, we see that

$$r = ks \in (0, 1), \qquad \frac{1}{p} = \frac{1-k}{p_0} + \frac{k}{2}.$$

Thus, by (2.3) we obtain

$$\|f\|_{H^r_p(\Omega)} \leqslant C_1 \|f\|^{1-k}_{L^{p_0}(\Omega)} \|f\|^k_{H^s(\Omega)},$$

for some constant $C_1 > 0$ and for all $f \in L^{p_0}(\Omega) \cap H^s(\Omega)$. □

**Corollary 2.2.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of cone-type. If $k, s \in (0, 1)$ and $p, p_0 > 1$ satisfy*

$$\frac{1}{p} = \frac{1-k}{p_0} + \frac{k}{2}, \tag{2.4}$$

*there exists $C_2 > 0$ such that,*

$$\|f\|_{L^p(\Omega)} \leqslant C_2 \|f\|^{1-k}_{L^{p_0}(\Omega)} \|f\|^k_{H^s(\Omega)},$$

*for all $f \in L^{p_0}(\Omega) \cap H^s(\Omega)$.*

**Proof.** First, it is clear that $r = ks \in (0,1)$. The Sobolev embedding $H_p^r(\Omega) \hookrightarrow L^p(\Omega)$, condition (2.4), and Lemma 2.1 give us

$$\|f\|_{L^p(\Omega)} \leqslant C_2 \|f\|_{L^{p_0}(\Omega)}^{1-k} \|f\|_{H^s(\Omega)}^k,$$

for some constant $C_2 > 0$. □

As a particular case of the Lemma 2.1 in the periodic context, we establish the following theorem.

**Proposition 2.3** *($n$-dimensional periodic Gagliardo-Nirenberg inequality). Let $\mathbb{T}^n \subset \mathbb{R}^n$ be the $n$-dimensional torus. If $k, s \in (0,1)$ and $r > 0$ are so that $r = ks$, then there exists $C_3 > 0$ such that*

$$\|f\|_{H_{per}^r(\mathbb{T}^n)} \leqslant C_3 \|f\|_{L_{per}^2(\mathbb{T}^n)}^{1-k} \|f\|_{H_{per}^s(\mathbb{T}^n)}^k, \tag{2.5}$$

*for all $f \in H_{per}^s(\mathbb{T}^n)$.*

**Proof.** Since the $n$-dimensional torus $\mathbb{T}^n \subset \mathbb{R}^n$ is a bounded domain of cone-type ([58, Section 4.2.3, Remark 5]), we obtain that the Lemma 2.1 is valid for $\Omega = \mathbb{T}^n$, $p_0 = 2$ and $r = ks \in (0,1)$. Moreover, by [58, Section 4.6.1, Equation 2] and [59, Section 9.1.3, Remark 1] the norms in $H_{per}^s(\Omega)$ and $H^s(\Omega)$ are equivalent and since $L^m(\mathbb{T}^n) \equiv L_{per}^m(\mathbb{T}^n)$, for all $m \geqslant 1$, it follows by (2.2) the following inequality

$$\|f\|_{H_{per}^r(\mathbb{T}^n)} \leqslant C_3 \|f\|_{L_{per}^2(\mathbb{T}^n)}^{1-k} \|f\|_{H_{per}^s(\mathbb{T}^n)}^k,$$

for all $f \in H_{per}^s(\mathbb{T}^n)$ and for some constant $C_3 > 0$. □

**Corollary 2.4** *(1-dimensional Periodic Gagliardo-Nirenberg inequality). Let $s \in \left(\frac{1}{4}, 1\right)$ be fixed. There exists a constant $C_4 > 0$ such that,*

$$\|f\|_{L_{per}^4}^4 \leqslant C_4 \|(-\Delta)^{\frac{s}{2}} f\|_{L_{per}^2}^{\frac{1}{s}} \|f\|_{L_{per}^2}^{4-\frac{1}{s}} + C_4 \|f\|_{L_{per}^2}^4, \tag{2.6}$$

*for all $f \in H_{per}^s$.*

**Proof.** In Proposition 2.3, let us consider $n = 1$. We have

$$\|f\|_{H_{per}^r} \leqslant C_3 \|f\|_{L_{per}^2}^{1-k} \|f\|_{H_{per}^s}^k,$$

for all $f \in H_{per}^s$. Here, we consider $r = ks \in (0,1)$, $k \in (0,1)$, and $s \in (0,1)$.

Several calculations and the definition of the norm of $H_{per}^s$ given by (1.17) yield the existence of a constant $C_5 > 0$ where

$$\|f\|_{H_{per}^r}^4 \leqslant C_5 \|f\|_{L_{per}^2}^{4(1-k)} \left( \|(-\Delta)^{\frac{s}{2}} f\|_{L_{per}^2}^2 + \|f\|_{L_{per}^2}^2 \right)^{2k},$$

for all $f \in H^s_{per}$. By [36, Lemma 3.197], we obtain the existence of a constant $C_6 > 0$ such that

$$\left(\|(-\Delta)^{\frac{s}{2}} f\|^2_{L^2_{per}} + \|f\|^2_{L^2_{per}}\right)^{2k} \leqslant C_6 \left(\|(-\Delta)^{\frac{s}{2}} f\|^{4k}_{L^2_{per}} + \|f\|^{4k}_{L^2_{per}}\right).$$

Thus, there exists a constant $C_7 > 0$ such that

$$\|f\|^4_{H^r_{per}} \leqslant C_7 \|f\|^{4(1-k)}_{L^2_{per}} \|(-\Delta)^{\frac{s}{2}} f\|^{4k}_{L^2_{per}} + C_7 \|f\|^4_{L^2_{per}}. \tag{2.7}$$

Choosing $r = \frac{1}{4}$ and using the embedding $H^r_{per} \hookrightarrow L^4_{per}$ (see [3, Theorem 4.2]), we obtain from (2.7) for $s = \frac{1}{4k} \in \left(\frac{1}{4}, 1\right)$ that

$$\|f\|^4_{L^4_{per}} \leqslant C_4 \|(-\Delta)^{\frac{s}{2}} f\|^{\frac{1}{s}}_{L^2_{per}} \|f\|^{4-\frac{1}{s}}_{L^2_{per}} + C_4 \|f\|^4_{L^2_{per}},$$

for some constant $C_4 > 0$. $\square$

The existence of global solutions in time for the Cauchy problem associated with the equation (1.1) is obtained by the combination of Corollary 2.4 and the conserved quantities $E$ and $F$ given by (1.3) and (1.4), respectively. In fact, as mentioned in the Introduction, we see that for $s \in \left(\frac{1}{2}, 1\right)$, there exists a local solution $u \in C([0, T], H^s_{per})$ of the Cauchy problem (2.1) associated with the equation (1.1) with initial data $u_0 \in H^s_{per}$ (see [20] and [15]). For all $t \geqslant 0$, we have

$$\|(-\Delta)^{\frac{s}{2}} u(t)\|^2_{L^2_{per}} = 2E(u_0) + \frac{1}{2}\|u(t)\|^4_{L^4_{per}}.$$

By Corollary 2.4 we obtain the existence of a constant $C > 0$ such that

$$\|(-\Delta)^{\frac{s}{2}} u(t)\|^2_{L^2_{per}} \leqslant 2E(u_0) + C\|u_0\|^{4-\frac{1}{s}}_{L^2_{per}} \|(-\Delta)^{\frac{s}{2}} u(t)\|^{\frac{1}{s}}_{L^2_{per}} + C\|u_0\|^4_{L^2_{per}}, \tag{2.8}$$

where we are using the fact that the $L^2_{per}$-norm is a conserved quantity (see (1.4)).

Therefore, by (2.8), we obtain the following scenario for global solutions $H^s_{per}$:

- When $s \in \left(\frac{1}{2}, 1\right]$, we can proceed similarly to the authors in [7] to conclude the existence of global solutions in time.
- When $s = \frac{1}{2}$, we use again [7] to conclude the existence of global solutions in time for $||u_0||_{L^2_{per}}$ small enough. It is also expected blow-up in finite time for large $||u_0||_{L^2_{per}}$.

Summarizing our analysis performed above, we obtain the following global well-posedness result for the Cauchy problem associated with the fNLS equation (1.1).

**Proposition 2.5.** *Let $s \in \left(\frac{1}{2}, 1\right]$. The Cauchy problem associated with the equation (1.1) is globally well-posed in $H^s_{per}$. More precisely, for any $u_0 \in H^s_{per}$ there exists a unique global solution $u \in C([0, +\infty), H^s_{per})$ such that $u(0) = u_0$ and it satisfies (1.1). Moreover, for each $T > 0$ the mapping*

$$u_0 \in H^s_{per} \longmapsto u \in C([0,T], H^s_{per})$$

*is continuous.*

## 3. Existence of periodic waves

In this section, we prove the existence of the even periodic wave solutions of (1.6) using two approaches. First, we use a variational characterization by minimizing a suitable constrained functional to obtain positive and even periodic waves with single-lobe profile. Second, we present some tools concerning the existence of small amplitude periodic waves using bifurcation theory. In addition, it is possible to show that such waves are also solutions for the minimization problem presented in the next subsection.

### 3.1. Existence of periodic waves via minimizers

In this subsection, we prove the existence of even periodic solutions for (1.6) by considering the variational problem given by (1.15). First, we define of the solution with *single-lobe* profile.

**Definition 3.1.** We say that a periodic wave satisfying the equation (1.6) has single-lobe profile if there exist only one maximum and minimum on $[-\pi, \pi]$. Without loss of generality, we assume that the maximum point occurs at $x = 0$.

For $\tau > 0$, let us consider the set

$$\mathcal{Y}_\tau := \left\{ u \in H^s_{per,e} \, ; \, \|u\|^4_{L^4_{per}} = \tau \right\}. \tag{3.1}$$

For $\omega > 0$, we define the functional $\mathcal{B}_\omega : H^s_{per,e} \longrightarrow \mathbb{R}$ given by

$$\mathcal{B}_\omega(u) := \frac{1}{2} \int_{-\pi}^{\pi} |(-\Delta)^{\frac{s}{2}} u|^2 + \omega |u|^2 \, dx, \tag{3.2}$$

for all $u \in H^s_{per,e}$.

We see that

$$\mathcal{B}_\omega(u) \geqslant 0 \quad \text{and} \quad G(u) \leqslant \mathcal{B}_\omega(u). \tag{3.3}$$

We have the following result of existence:

**Proposition 3.2.** *Let $s \in \left(\frac{1}{4}, 1\right]$ and $\tau, \omega > 0$ be fixed. The minimization problem*

$$\Gamma_\omega := \inf_{u \in \mathcal{Y}_\tau} \mathcal{B}_\omega(u) \tag{3.4}$$

*has at least one solution, that is, there exists a complex-valued function $\Phi \in \mathcal{Y}_\tau$ such that $\mathcal{B}_\omega(\Phi) = \Gamma_\omega$. Moreover, $\Phi$ satisfies*

$$(-\Delta)^s \Phi + \omega \Phi - |\Phi|^2 \Phi = 0.$$

**Proof.** First we claim that the functional $\mathcal{B}_\omega$ induces an equivalent norm in $H^s_{per,e}$ which is the induced norm of $H^s_{per}$. Indeed, the norm in $H^s_{per}$ is given as in (1.17) and the functional $\mathcal{B}_\omega$ can be written as $2\mathcal{B}_\omega(u) = \|(-\Delta)^{\frac{s}{2}}u\|^2_{L^2_{per}} + \omega\|u\|^2_{L^2_{per}}$. Thus, it is easy to see that there exist constants $c_0, c_1 > 0$ so that

$$0 \leqslant c_0\|u\|_{H^s_{per}} \leqslant \sqrt{2\mathcal{B}_\omega(u)} \leqslant c_1\|u\|_{H^s_{per}}. \tag{3.5}$$

Moreover, by (3.3), one has $\Gamma_\omega \geqslant 0$.

Using the smoothness of the functional $\mathcal{B}_\omega$, we may consider a sequence of minimizers $(u_n)_{n\in\mathbb{N}} \subset \mathcal{Y}_\tau$ such that

$$\mathcal{B}_\omega(u_n) \longrightarrow \Gamma_\omega, \quad n \to \infty. \tag{3.6}$$

By (3.6), we have that the sequence $(\mathcal{B}_\omega(u_n))_{n\in\mathbb{N}} \subset \mathbb{R}$ is bounded, so that it is bounded in $H^s_{per,e}$. Since $s \in \left(\frac{1}{4}, 1\right]$ and the Sobolev space $H^s_{per,e}$ is reflexive, there exists $\Phi \in H^s_{per,e}$ such that (modulus a subsequence),

$$u_n \rightharpoonup \Phi \text{weakly in } H^s_{per,e}. \tag{3.7}$$

Again, since $s \in \left(\frac{1}{4}, 1\right]$, we obtain that the embedding

$$H^s_{per,e} \hookrightarrow L^4_{per} \tag{3.8}$$

is compact (see [5, Theorem 2.8] or [1, Theorem 5.1]). Thus, modulus a subsequence we also have

$$u_n \longrightarrow \Phi \text{ in } L^4_{per}. \tag{3.9}$$

Moreover, using the estimate

$$\begin{aligned}
&\left|\int_{-\pi}^{\pi} \left(|u_n|^4 - |\Phi|^4\right) dx\right| \\
&\quad \leqslant \int_{-\pi}^{\pi} \left||u_n|^4 - |\Phi^4|\right| dx \\
&\quad \leqslant \left(\|\Phi\|^3_{L^4_{per}} + \|\Phi\|^2_{L^4_{per}}\|u_n\|_{L^4_{per}} + \|\Phi\|_{L^4_{per}}\|u_n\|^2_{L^4_{per}} + \|u_n\|^3_{L^4_{per}}\right)\|u_n - \Phi\|_{L^4_{per}}
\end{aligned}$$

and (3.9), it follows that $\|\Phi\|^4_{L^4_{per}} = \tau$. Furthermore, since $\mathcal{B}_\omega$ is lower semi-continuous, we have

$$\mathcal{B}_\omega(\Phi) \leqslant \liminf_{n\to\infty} \mathcal{B}_\omega(u_n)$$

that is,

$$\mathcal{B}_\omega(\Phi) \leqslant \Gamma_\omega. \tag{3.10}$$

On the other hand, once $\Phi$ satisfies $\|\Phi\|^4_{L^4_{per}} = \tau$, we obtain

$$\mathcal{B}_\omega(\Phi) \geqslant \Gamma_\omega. \tag{3.11}$$

By (3.10) and (3.11) we conclude

$$\mathcal{B}_\omega(\Phi) = \Gamma_\omega = \inf_{u \in \mathcal{Y}_\tau} \mathcal{B}_\omega(u).$$

In other words, the function $\Phi \in \mathcal{Y}_\tau$ is a minimizer of the problem (3.4). Note that since $\tau > 0$, we see that $\Phi$ is a complex-valued function such that $\Phi \not\equiv 0$. In addition, as a consequence of the Lagrange multiplier theorem, there exists a constant $c_2 = \frac{2B_\omega(\Phi)}{\tau} > 0$, so that

$$(-\Delta)^s \Phi + \omega\Phi = c_2|\Phi|^2\Phi. \tag{3.12}$$

A scaling argument as $\Psi \equiv \sqrt{c_2}\Phi$ allows us to choose $c_2 = 1$ in (3.12) (see similar arguments in [17, page 629]). Thus, we have that $\Phi$ is a periodic minimizer of the problem (1.15) and it satisfies the equation

$$(-\Delta)^s \Phi + \omega\Phi - |\Phi|^2\Phi = 0. \quad \square \tag{3.13}$$

**Remark 3.3.** Let $\Phi \in H^s_{per}$ be the minimizer obtained by Proposition 3.2. It is easy to check that for all $\theta \in \mathbb{R}$, function $e^{-i\theta}\Phi$ satisfies $\mathcal{B}_\omega(e^{-i\theta}\Phi) = \Gamma_\omega$ and consequently (3.13). To guarantee the existence of real-valued solutions for the equation (3.13), we assume that the minimizer $\Phi$ can be expressed as $\Phi = e^{i\theta_0}\varphi$, where $\varphi$ is a real $2\pi$-periodic function and $\theta_0$ is a suitable real number. Function $\varphi \in H^s_{per}$ satisfies (1.6) and the minimization problem

$$\mathcal{B}_\omega(\varphi) = \mathcal{B}_\omega(e^{-i\theta_0}\Phi) = \Gamma_\omega. \tag{3.14}$$

**Remark 3.4.** In the particular case $s = 1$, the existence of $\theta_0$ and $\varphi$ in Remark 3.3 can be established. In fact, if $\Phi = \phi_1 + i\phi_2$ solves (3.13), we see that $\phi_1$ and $\phi_2$ solve the equations

$$-\phi_1'' + \omega\phi_1 - (\phi_1^2 + \phi_2^2)\phi_1 = 0, \tag{3.15}$$

and

$$-\phi_2'' + \omega\phi_2 - (\phi_1^2 + \phi_2^2)\phi_2 = 0. \tag{3.16}$$

Multiplying (3.15) by $\phi_2$, (3.16) by $\phi_1$ and subtracting both results we obtain the equation $-\phi_1''\phi_2 + \phi_2''\phi_1 = 0$, so that $-\phi_1'\phi_2 + \phi_2'\phi_1 = C$, where $C$ is a constant of integration. Since $\phi_1$ and $\phi_2$ are even, we obtain $C = 0$ because $\phi_1'$ and $\phi_2'$ are odd. Equation $-\phi_1'\phi_2 + \phi_2'\phi_1 = 0$ thus implies $\phi_1 = r\phi_2$ for some $r \in \mathbb{R}$, so that $\Phi = (r + i)\phi_2 = e^{i\theta_0}\sqrt{1 + r^2}\phi_2$, where $\theta_0$ is the principal argument of the complex number $r + i$. Therefore for $\varphi = \sqrt{1 + r^2}\phi_2$, one has the desired result.

**Remark 3.5.** Let $s \in (0, 1)$ and $\omega > 0$ be fixed. It is worth to be mentioning that if the problem (3.4) is posed in the infinite wavelength scenario, that is, if $\Phi \in H^s(\mathbb{R})$ is a minimizer, then $|\Phi|$ is also a minimizer for the functional $\widetilde{\mathcal{B}}_\omega(u) = \frac{1}{2}\int_{\mathbb{R}} |(-\Delta)^{\frac{s}{2}}u|^2 + \omega|u|^2 dx = \frac{1}{2}||(-\Delta)^{\frac{s}{2}}u||_{L^2}^2 + \frac{\omega}{2}||u||_{L^2}^2$, with $u \in H^s(\mathbb{R})$ and satisfying $\int_{\mathbb{R}} |u|^4 dx = \tau$. This fact can be determined since the semi-norm $||(-\Delta)^{\frac{s}{2}}u||_{L^2}^2$ can be characterized in terms of the well-known *Gagliardo semi-norm*

$$||(-\Delta)^{\frac{s}{2}}u||_{L^2} = C_s \left( \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(x) - u(y)|^2}{|x - y|^{1+2s}} dxdy \right)^{\frac{1}{2}}, \tag{3.17}$$

where $C_s$ is a positive constant depending only on $s \in (0, 1)$. By (3.17), it is possible to deduce that $|\Phi|$ is a minimizer if $\Phi$ is, and thus a real-valued solution $\varphi := |\Phi|$ for the equation (1.6) can be considered without further problems (for more details, see inequality (25) in [32, page 3454]). As far as we know, we do not have a characterization as in (3.17) for the periodic case. Indeed, it is possible to define the Gagliardo semi-norm in the periodic context as

$$[u]_s = \left( \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|u(x) - u(y)|^2}{|x - y|^{1+2s}} dxdy \right)^{\frac{1}{2}}. \tag{3.18}$$

However, using Parseval's identity and some additional calculations (see [3, page 8]) we obtain the existence of $c_3 > c_2 > 0$ such that

$$c_2||(-\Delta)^{\frac{s}{2}}u||_{L^2_{per}} \leqslant [u]_s \leqslant c_3||(-\Delta)^{\frac{s}{2}}u||_{L^2_{per}}. \tag{3.19}$$

Inequality in (3.19) can not be used to assure that $|\Phi|$ solves (3.4) if $\Phi$ is a periodic minimizer obtained by Proposition 3.2. In our context, assumption $\Phi = e^{i\theta_0}\varphi$ in Remark 3.3 seems suitable necessary in order to obtain a real-valued periodic solution $\varphi = \varphi + i0$ for the problem (3.4).

As a consequence of the assumption in Remark 3.3, we have the following result.

**Proposition 3.6** *(Existence of Even Single-Lobe Solutions). Let $s \in \left(\frac{1}{4}, 1\right]$ and $\omega > 0$ be fixed. Let $\varphi \in H^s_{per}$ be the real-valued periodic minimizer given by (3.14). If $\omega \in \left(0, \frac{1}{2}\right]$ then $\varphi$ is the constant solution and if $\omega \in \left(\frac{1}{2}, +\infty\right)$ then $\varphi$ is an even periodic single-lobe solution for the equation (1.6).*

**Proof.** First, by a bootstrapping argument we infer that $\varphi \in H^\infty_{per,e}$ (see [17, Proposition 3.1] and [44, Proposition 2.4]).

Since the solution can be constant, we need to avoid this case in order to guarantee that the minimizer has a single-lobe profile. First, we see that the positive constant solution of the equation (1.6) is $\varphi \equiv \sqrt{\omega}$ and the operator $\mathcal{L}_1$ in (1.9) is then given by $\mathcal{L}_1 = (-\Delta)^s - 2\omega$. By [34, Example 4.4] we obtain that $\mathrm{n}(\mathcal{L}_1) = 1$ if and only if $\omega \in \left(0, \frac{1}{2}\right]$. In addition, we have to notice that $\varphi = \sqrt{\omega}$ is not a minimizer of (3.4) for $\omega > \frac{1}{2}$ since in this case we have $\mathrm{n}(\mathcal{L}_1) > 1$ (for $\omega > \frac{1}{2}$ we see that $\varphi$ is a periodic minimizer of $G$ restricted only to one constraint and it is expected that $n(\mathcal{L}_1) \leqslant 1$ since $n(\mathcal{L}) \leqslant 1$). In addition, we will see in Section 4 that if $\varphi$ is a nonconstant minimizer then $\mathcal{L}_1\varphi' = 0$ which implies, by Sturm's oscillation theorem, in fact that $\mathrm{n}(\mathcal{L}_1) = 1$. Thus, we conclude that the constant solution $\varphi = \sqrt{\omega}$ is a minimizer of (3.4) for $\omega \in \left(0, \frac{1}{2}\right]$ and for $\omega \in \left(\frac{1}{2}, +\infty\right)$, solution $\varphi$ is a nonconstant minimizer. Furthermore, in the latter case, we can consider the symmetric rearrangements $\varphi^\star$ associated with $\varphi$ and it is well known that such rearrangements are invariant under the constraint of $\mathcal{Y}_\tau$ by using [16, Appendix A]. Moreover, due to the fractional Polya-Szegö inequality, in [16, Lemma A.1], we have

$$\int_{-\pi}^{\pi} \left((-\Delta)^{\frac{s}{2}}\varphi^\star\right)^2 dx \leqslant \int_{-\pi}^{\pi} \left((-\Delta)^{\frac{s}{2}}\varphi\right)^2 dx.$$

Thus, by (3.14), we obtain $\mathcal{B}_\omega(\varphi^\star) = \Gamma_\omega$ with $\varphi^\star$ being symmetrically decreasing away from the maximum point $x = 0$. To simplify the notation, we assume that $\varphi = \varphi^\star$, so that $\varphi$ has an even single-lobe profile according to the Definition 3.1. □

### 3.2. Small-amplitude periodic waves

The existence and convenient formulas for the small amplitude periodic waves associated with the equation (1.6) will be shown in this subsection. After that, we show that the local bifurcation theory used to determine the existence of small amplitude waves can be extended and the local solutions can be considered as global for a fixed $\omega > \frac{1}{2}$. This fact is a very important feature in our context since it can be used as an alternative form to prove the existence of periodic even solutions (not necessarily having a single-lobe profile) for the equation (1.6). To do so, we use the theory contained in [11, Chapters 8 and 9].

First, we shall give some steps to prove the existence of small amplitude periodic waves. In fact, for $s \in (\frac{1}{4}, 1]$, let $\mathsf{F} : H^{2s}_{per,e} \times (\frac{1}{2}, +\infty) \to L^2_{per,e}$ be the smooth map defined by

$$\mathsf{F}(g, \omega) = (-\Delta)^s g + \omega g - g^3. \tag{3.20}$$

We see that $\mathsf{F}(g, \omega) = 0$ if and only if $g \in H^{2s}_{per,e}$ satisfies (1.6) with corresponding wave frequency $\omega \in (\frac{1}{2}, +\infty)$. The Fréchet derivative of the function $\mathsf{F}$ with respect to the first variable is then given by

$$D_g\mathsf{F}(g, \omega) f = \left((-\Delta)^s + \omega - 3g^2\right) f. \tag{3.21}$$

Let $\omega_0 > \frac{1}{2}$ be fixed. At the point $(\sqrt{\omega_0}, \omega_0)$, we have that

$$D_g\mathsf{F}(\sqrt{\omega_0}, \omega_0) = (-\Delta)^s + \omega_0 - 3(\sqrt{\omega_0})^2 = (-\Delta)^s - 2\omega_0. \tag{3.22}$$

The nontrivial kernel of $D_g\mathsf{F}(\sqrt{\omega_0}, \omega_0)$ is determined by functions $h \in H^{2s}_{per,e}$ such that

$$\widehat{h}(k)(-2\omega_0 + |k|^{2s}) = 0. \tag{3.23}$$

We see that $D_g\mathsf{F}(\sqrt{\omega_0}, \omega_0)$ has the one-dimensional kernel if and only if $\omega_0 = \frac{|k|^{2s}}{2}$ for some $k \in \mathbb{Z}$. In this case, we have

$$\mathrm{Ker} D_g \mathsf{F}(\sqrt{\omega_0}, \omega_0) = [\tilde{\varphi}_k], \tag{3.24}$$

where $\tilde{\varphi}_k(x) = \cos(kx)$.

The local bifurcation theory contained in [11, Chapter 8] enables us to guarantee the existence of an open interval $I$ containing $\omega_0 > \frac{1}{2}$, an open ball $B(0,r) \subset H^{2s}_{per,e}$ for some $r > 0$ and a unique smooth mapping

$$\omega \in I \longmapsto \varphi := \varphi_\omega \in B(0,r) \subset H^{2s}_{per,e}$$

such that $\mathsf{F}(\varphi, \omega) = 0$ for all $\omega \in I$ and $\varphi \in B(0,r)$.

For each $k \in \mathbb{N}$, the point $(\sqrt{\tilde{\omega}_k}, \tilde{\omega}_k)$ where $\tilde{\omega}_k := \frac{|k|^{2s}}{2}$ is a bifurcation point. Moreover, there exists $a_0 > 0$ and a local bifurcation curve

$$a \in (0, a_0) \longmapsto (\varphi_{k,a}, \omega_{k,a}) \in H^{2s}_{per,e} \times (0, +\infty) \tag{3.25}$$

which emanates from the point $(\sqrt{\tilde{\omega}_k}, \tilde{\omega}_k)$ to obtain small amplitude even $\frac{2\pi}{k}$-periodic solutions for the equation (1.6). In addition, we have $\omega_{k,0} = \tilde{\omega}_k$, $D_a\varphi_{k,0} = \tilde{\varphi}_k$ and all solutions of $\mathsf{F}(g,\omega) = 0$ in a neighbourhood of $(\sqrt{\tilde{\omega}_k}, \tilde{\omega}_k)$ belong to the curve in (3.25) depending on $a \in (0, a_0)$.

**Proposition 3.7.** *Let $s \in (0,1]$ be fixed. There exists $a_0 > 0$ such that for all $a \in (0, a_0)$ there is a unique even local periodic solution $\varphi$ for the problem (1.6). The small amplitude periodic waves are given by the following expansion:*

$$\varphi(x) = \sqrt{\omega} + \sqrt{2}\phi(x), \tag{3.26}$$

*where*

$$\phi(x) = a\phi_1(x) + a^2\phi_2(x) + a^3\phi_3(x) + \mathcal{O}(a^4). \tag{3.27}$$

*Here $\phi_1(x) = \cos(x)$,*

$$\phi_2(x) = -\frac{3}{2} + \frac{3}{2(2^{2s}-1)}\cos(2x),$$
$$\phi_3(x) = \frac{1}{2(3^{2s}-1)}\left[1 + \frac{9}{2^{2s}-1}\right]\cos(3x),$$

*and*

$$\gamma = \frac{15}{2} - \frac{9}{2(2^{2s}-1)}.$$

*The frequency $\omega$ in this case is expressed as*

$$\omega = \frac{1}{2} + a^2\gamma + \mathcal{O}(a^4). \tag{3.28}$$

*For $s \in (\frac{1}{4}, 1]$, the pair $(\varphi, \omega) \in H^s_{per,e} \times (\frac{1}{2}, +\infty)$ is global in terms of the parameter $\omega > \frac{1}{2}$ and it satisfies (1.6).*

**Proof.** The first part of the proposition has been already determined in (3.25) by considering $k = 1$. To get the expression in (3.26), we use arguments similar to the ones in [44, Section 5]. To obtain that the local curve (3.25) extends to a global one for the case $s \in (\frac{1}{4}, 1]$, we first need to prove that $D_g\mathsf{F}(g, \omega)$ given by (3.21) is a Fredholm operator of index zero. Indeed, we define the set $\mathcal{S} = \{(g, \omega) \in D(\mathsf{F}) : \mathsf{F}(g, \omega) = 0\}$. Let $(g, \omega) \in H^{2s}_{per,e} \times (\frac{1}{2}, +\infty)$ be a real solution of $\mathsf{F}(g, \omega) = 0$. For $\mathcal{Z} := L^2_{per,e}$, the linear operator

$$\mathcal{L}_{1|_\mathcal{Z}}\psi \equiv D_g\mathsf{F}(g, \omega)\psi = \left((-\Delta)^s - 3g^2\right)\psi + \omega\psi = 0, \tag{3.29}$$

has two linearly independent solutions and at most one belongs to $H^{2s}_{per,e}$ (see [16, Theorem 3.12]). If there are no solutions in $H^{2s}_{per,e}\backslash\{0\}$, then the problem $\left((-\Delta)^s + \omega - 3g^2\right)\psi = f$ has a unique non-trivial solution $\psi \in H^{2s}_{per,e}$ for all $f \in \mathcal{Z}$ since $\mathrm{Ker}(\mathcal{L}_{1|_\mathcal{Z}})^\perp = \mathrm{R}(\mathcal{L}_{1|_\mathcal{Z}}) = \mathcal{Z}$.

On the other hand, if there is a solution $e \in H^{2s}_{per,e}$ we obtain by standard Fredholm Alternative that (3.29) has a solution if and only if

$$\int_{-\pi}^{\pi} e(x)f(x)dx = 0,$$

for all $f \in \mathcal{Y}$. We can conclude in both cases that the Fréchet derivative of $\mathsf{F}$ in terms of $g$ given by (3.21) is a Fredholm operator of index zero.

Let us prove that every bounded and closed $\mathcal{S}$ is a compact set on $H^{2s}_{per,e} \times (\frac{1}{2}, +\infty)$. For $g \in H^{2s}_{per,e}$ and $\omega > \frac{1}{2}$, we define $\widetilde{\mathsf{F}}(g, \omega) = ((-\Delta)^s + \omega)^{-1}g^3$. Since $s \in (\frac{1}{4}, 1]$, we see that $\widetilde{\mathsf{F}}$ is well defined since $H^{2s}_{per,e}$ is a Banach algebra, $(g, \omega) \in \mathcal{S}$ if and only if $\widetilde{\mathsf{F}}(g, \omega) = g$ and $\widetilde{\mathsf{F}}$ maps $H^{2s}_{per,e} \times (\frac{1}{2}, +\infty)$ into $H^{4s}_{per,e}$. The compact embedding $H^{4s}_{per,e} \hookrightarrow H^{2s}_{per,e}$ shows that $\widetilde{\mathsf{F}}$ maps bounded and closed sets in $H^{2s}_{per,e} \times (\frac{1}{2}, +\infty)$ into $H^{2s}_{per,e}$. Thus, if $\mathcal{R} \subset \mathcal{S} \subset H^{2s}_{per,e} \times (\frac{1}{2}, +\infty)$ is a bounded and closed set, we obtain that $\widetilde{\mathsf{F}}(\mathcal{R})$ is relatively compact in $H^{2s}_{per,e}$. Since $\mathcal{R}$ is closed, any sequence $\{(\varphi_n, \omega_n)\}_{n\in\mathbb{N}}$ has a convergent subsequence in $\mathcal{R}$, so $\mathcal{R}$ is compact in $H^{2s}_{per,e} \times (\frac{1}{2}, +\infty)$.

Since the frequency of the wave given by (3.28) is not constant, we can apply [11, Theorem 9.1.1] to extend globally the local bifurcation curve given in (3.25). More precisely, there is a continuous mapping

$$\omega \in \left(\tfrac{1}{2}, +\infty\right) \longmapsto \varphi_\omega \in H^{2s}_{per,e} \tag{3.30}$$

where $\varphi_\omega$ solves equation (1.6). □

**Remark 3.8.** It is important to note that $\varphi \in H^{2s}_{per,e}$ given by (3.26) is a solution of the minimization problem (3.4) by using similar arguments as in [45, Lemma 2.3].

## 4. Spectral analysis and uniqueness of minimizers

### 4.1. Spectral analysis

We use the variational characterization determined in the last section to obtain useful spectral properties for the linearized operator $\mathcal{L}$ in (1.8) around the periodic wave $\varphi$ obtained by Proposition 3.6.

Let $s \in \left(\frac{1}{4}, 1\right]$ and $\omega > 0$ be fixed. Consider the periodic minimizer $\varphi = \varphi_\omega \in H^\infty_{per,e}$ obtained by Proposition 3.6. We study the spectral properties of the matrix operator

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_1 & 0 \\ 0 & \mathcal{L}_2 \end{pmatrix} : H^{2s}_{per} \times H^{2s}_{per} \subset L^2_{per} \times L^2_{per} \to L^2_{per} \times L^2_{per},$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are the real and imaginary parts of the operator $\mathcal{L}$ and they are defined by

$$\mathcal{L}_1 = (-\Delta)^s + \omega - 3\varphi^2 \qquad \text{and} \qquad \mathcal{L}_2 = (-\Delta)^s + \omega - \varphi^2. \tag{4.1}$$

Thanks to the variational formulation (3.4), Proposition 3.2 and Remark 3.3, we obtain $\varphi$ as a minimizer of $G(u)$ in (1.7) for every $\omega > \frac{1}{2}$ subject to one constraint. In the even sector of $L^2_{per}$, since $\mathcal{L}$ in (1.8) is the Hessian operator for $G(u)$ in (1.7), it follows by the min-max principle [53, Theorem XIII.2] that $\mathrm{n}(\mathcal{L}) \leqslant 1$. Since $\mathcal{L}_1\varphi = -2\varphi^3$ and $(\mathcal{L}_1\varphi, \varphi)_{L^2_{per}} = -2\int_{-\pi}^{\pi} \varphi^4 dx < 0$, we have $\mathrm{n}(\mathcal{L}_1) \geqslant 1$. The operator $\mathcal{L}$ in (1.8) is diagonal and thus $\mathrm{n}(\mathcal{L}_1) = \mathrm{n}(\mathcal{L}) = 1$, so that $\mathrm{n}(\mathcal{L}_2) = 0$. Next, we see that $\mathcal{L}_2\varphi = 0$ with $\mathrm{n}(\mathcal{L}_2) = 0$. It follows by oscillation theorem in [34] that $\varphi > 0$ and the zero eigenvalue for $\mathcal{L}_2$ results to be simple in the even sector of $L^2_{per}$. By [33, Lemma 3.3], we obtain that zero is the first eigenvalue of $\mathcal{L}_1$ whose associated eigenfunction is $\varphi'$ in the odd sector of $L^2_{per}$. Therefore, in the whole space $L^2_{per}$ we obtain $\mathrm{n}(\mathcal{L}_1) = 1$, so that $\mathrm{n}(\mathcal{L}) = 1$ in $L^2_{per} \times L^2_{per}$. In fact, we have proved the following result.

**Lemma 4.1.** *Let $s \in \left(\frac{1}{4}, 1\right]$ and $\omega > \frac{1}{2}$ be fixed. If $\varphi \in H^\infty_{per,e}$ is the periodic minimizer given by Proposition 3.6, then $\mathrm{n}(\mathcal{L}_2) = 0$ and $\mathrm{z}(\mathcal{L}_2) = 1$.*

Concerning the operator $\mathcal{L}_1$ in (4.1), we have the following lemma.

**Lemma 4.2.** *Let $s \in \left(\frac{1}{4}, 1\right]$ and $\omega > \frac{1}{2}$ be fixed. If $\varphi \in H^\infty_{per,e}$ is the periodic minimizer given by Proposition 3.6 and $\omega \in \left(\frac{1}{2}, +\infty\right) \mapsto \varphi$ is smooth, then $\mathrm{Ker}(\mathcal{L}_1) = [\varphi']$.*

**Proof.** First, we see that $\varphi' \in \mathrm{Ker}(\mathcal{L}_1) = \mathrm{R}(\mathcal{L}_1)^\perp$. In addition, since $\mathcal{L}_1\varphi = -2\varphi^3$, we obtain $\varphi^3 \in \mathrm{R}(\mathcal{L}_1)$. On the other hand, differentiating equation (1.6) with respect to $\omega$ we obtain $\mathcal{L}_1\left(-\frac{d}{d\omega}\varphi\right) = \varphi$, so that $\varphi \in \mathrm{R}(\mathcal{L}_1)$.

Arguments above establish in fact that $\varphi, \varphi^3 \in \mathrm{R}(\mathcal{L}_1) = \mathrm{Ker}(\mathcal{L}_1)^\perp$ with $\varphi$ being an even, smooth, positive, and single-lobe solution for (1.6). Let us assume that $\mathrm{z}(\mathcal{L}_1) = 2$. Since $\varphi' \in \mathrm{Ker}(\mathcal{L}_1)$ is odd, there exists an even periodic function $h \in \mathrm{Ker}(\mathcal{L}_1)$ such that $h$ has exactly two symmetric zeros in the interval $[-\pi, \pi)$ (see oscillation theorem in [34]). Hence, there exists $x_0 \in (-\pi, \pi)$ such that $h(\pm x_0) = 0$. Without loss of generality, we can still suppose that

$$h(x) > 0,\ x \in (-x_0, x_0) \quad \text{and} \quad h(x) < 0,\ x \in [-\pi, x_0) \cup (x_0, \pi). \tag{4.2}$$

Furthermore, since $h \in \mathrm{Ker}(\mathcal{L}_1)$ and $\varphi, \varphi^3 \in \mathrm{Ker}(\mathcal{L}_1)^{\perp}$ we have

$$(h, \varphi^3)_{L^2_{per}} = 0 \quad \text{and} \quad (h, \varphi)_{L^2_{per}} = 0. \tag{4.3}$$

Since $\varphi > 0$, we obtain by the fact that $\varphi$ is a single-lobe that $\varphi(x)(\varphi(x)^2 - \varphi^2(x_0))$ is positive over $(-x_0, x_0)$ and negative over $[-\pi, x_0) \cup (x_0, \pi)$, so that it has the same behaviour as $h$ in (4.2). Thus, $(\varphi(\varphi^2 - \varphi^2(x_0)), h)_{L^2_{per}} \neq 0$ which leads a contradiction with (4.3). Consequently, we have $\mathrm{Ker}(\mathcal{L}_1) = [\varphi']$. □

**Remark 4.3.** Arguments established in the end of the proof of Lemma 4.2 are valid only if $\varphi > 0$. If $\varphi$ changes its sign over $\mathbb{T}$, an alternative form to prove that $\mathrm{Ker}(\mathcal{L}_1) = [\varphi']$ can be determined by proving that $1 \in \mathrm{R}(\mathcal{L}_1)$. In the affirmative case and since $\mathcal{L}_1 1 = \omega - 3\varphi^2$, we obtain that the property $\{1, \varphi, \varphi^2\} \subset \mathrm{R}(\mathcal{L}_1)$ occurs. Employing the arguments in [45, Proposition 2.5], we obtain that $\mathrm{Ker}(\mathcal{L}_1) = [\varphi']$ as requested.

A converse of Lemma 4.2 can be determined.

**Proposition 4.4.** *Let $s \in \left(\frac{1}{4}, 1\right]$ and $\varphi_0 \in H^{\infty}_{per}$ be the solution obtained in the Proposition 3.6 which is associated with the fixed value $\omega_0 > \frac{1}{2}$. If $\mathrm{Ker}(\mathcal{L}_1) = [\varphi_0']$, then there exists a $C^1$ mapping*

$$\omega \in \mathcal{I}_{\omega_0} \longmapsto \varphi_\omega \in H^s_{per,e}$$

*defined in an open neighbourhood $\mathcal{I}_{\omega_0} \subset (0, +\infty)$ of $\omega_0 > \frac{1}{2}$ such that $\varphi_{\omega_0} = \varphi_0$.*

**Proof.** The proof follows from the implicit function theorem and it is similar to [17, Theorem 3.2]. □

**Remark 4.5.** We cannot guarantee that for each $\omega \in \mathcal{I}_{\omega_0}$ given by Proposition 4.4 that $\varphi_\omega$ solves the minimization problem (3.4) except at $\omega = \omega_0$.

The results determined in this subsection can be summarized in the following proposition:

**Proposition 4.6.** *Let $\varphi$ be the single-lobe profile obtained in Proposition 3.6. If $\omega \in \left(\frac{1}{2}, +\infty\right) \mapsto \varphi$ is smooth, we have that $n(\mathcal{L}) = 1$ and $\mathrm{Ker}(\mathcal{L}) = [(\varphi', 0), (0, \varphi)]$.* ■

**Remark 4.7.** Let $s \in \left(\frac{1}{4}, 1\right]$ be fixed. The existence of small amplitude periodic waves established in Proposition 3.7 is smooth with respect to $\omega$ in a neighbourhood on the left side of the bifurcation point $\omega = \frac{1}{2}$. Therefore, at least inside this neighbourhood, we have that $\mathrm{Ker}(\mathcal{L}_1) = [\varphi']$ as requested in Lemma 4.2.

**Remark 4.8.** Let $s \in \left(\frac{1}{4}, 1\right]$ be fixed. Consider the change of variables $\varphi = \mathrm{m} + \psi$, where $\mathrm{m} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(x)dx$. We have $\int_{-\pi}^{\pi} \psi(x)dx = 0$ and since $\varphi$ solves (1.6), we obtain that $\psi$ solves the Gardner type-equation

$$(-\Delta)^s \psi + c\psi - 3\mathrm{m}\psi^2 - \psi^3 + b = 0, \tag{4.4}$$

where $c = \omega - 3\mathrm{m}^2$ and $b = \omega\mathrm{m} - \mathrm{m}^3$. Using the arguments determined in [45] (see Sections 4 and 5), we see that it is possible to construct, since $\mathrm{n}(\mathcal{L}_1) = 1$ for all $\omega \in \left(\frac{1}{2}, +\infty\right)$, a smooth surface $(c, \mathrm{m}) \in \mathcal{O} \mapsto \psi_{(c,\mathrm{m})} \in H^\infty_{per}$ of even periodic waves satisfying $\int_{-\pi}^{\pi} \psi_{(c,\mathrm{m})}(x)dx = 0$ for all $(c, \mathrm{m}) \in \mathcal{O}$. Here $\mathcal{O}$ is a convenient open subset of $\mathbb{R}^2$. The smooth surface of periodic waves $\psi$ allows us to deduce $\mathcal{L}_1(1 + \partial_\mathrm{m}\psi - 6\mathrm{m}\partial_c\psi) = c - \partial_\mathrm{m} b + 6\mathrm{m}\partial_c b := d$ and Lemma 4.7 in [44] gives us $\mathrm{Ker}(\mathcal{L}_1) = [\varphi']$ if and only if $d \neq 0$. Important to mention that numerical experiments contained in [44, Section 5] give us $d \neq 0$ for all $\omega \in \left(\frac{1}{2}, +\infty\right)$ and this fact seems reasonable since $\mathrm{n}(\mathcal{L}_1) = 1$ for all $\omega \in \left(\frac{1}{2}, +\infty\right)$. Indeed, if $\omega^*$ is the minimum value in $\left(\frac{1}{2}, +\infty\right)$ such that the kernel of $\mathcal{L}_1$ is double ($\omega^*$ is then called a *fold point*), it is expected to obtain the existence of $\omega_1 > \omega^*$ such that $\mathrm{n}(\mathcal{L}_1) = 2$ at $\omega = \omega_1$, which is a contradiction.

### 4.2. *Uniqueness of real minimizers*

In this subsection we show the uniqueness for the real periodic minimizers $\varphi$ obtained in Proposition 3.2. To this end, we proceed as in [2, Section 3.2]. The main difference in our approach is that we do not need to assume that the kernel of the linearized operator restricted to the space of zero mean periodic waves are simple. First, the space of zero mean periodic waves is not suitable for our purposes since we are working with real positive periodic waves $\varphi$. The equivalent condition in our case would be assuming that $\mathrm{Ker}(\mathcal{L}_1) = [\varphi']$ for every $\omega \in \left(\frac{1}{2}, +\infty\right)$ and $s \in \left(\frac{1}{4}, 1\right]$. In the remainder of this section, we assume only that $s \in \left(\frac{1}{4}, 1\right)$. The case $s = 1$ is not relevant in our analysis since the periodic (dnoidal) waves are unique for a fixed $\omega \in \left(\frac{1}{2}, +\infty\right)$.

In what follows, let us define the complex Banach space

$$\mathbb{V} := \{f = f_1 + i f_2 \equiv (f_1, f_2) \in L^4_{per} \times L^4_{per}\ ;\ f_1, f_2 \in L^4_{per,e}\},$$

endowed with the norm of $L^4_{per}$. We have the following result:

**Proposition 4.9.** *Let $s_0 \in (\frac{1}{2}, 1)$. Suppose that $(\varphi_0 + 0i, \omega_0) \in \mathbb{V} \times \left(\frac{1}{2}, +\infty\right)$ where $\varphi_0$ is a non-zero real solution of (1.6) with $s = s_0$ and $\omega = \omega_0$. If $\mathrm{Ker}(\mathcal{L}_1) = [\varphi_0']$, then for some $\delta > 0$, there exists a $C^1$-map $s \in I \to (\varphi_s + 0i, \omega_s) \in \mathbb{V} \times \left(\frac{1}{2}, +\infty\right)$, defined in the interval $I = [s_0, s_0 + \delta)$, such that the following holds:*

- (i) *$(\varphi_s + 0i, \omega_s)$ solves the equation (1.6) with $\omega = \omega_s$, for all $s \in I$;*
- (ii) *There exists $\varepsilon > 0$ such that $(\varphi_s + 0i, \omega_s)$ is the unique solution of (1.6) for $s \in I$ in the neighbourhood $\{(\varphi + 0i, \omega) \in \mathbb{V} \times \left(\frac{1}{2}, +\infty\right);\ ||\varphi - \varphi_0||_{\mathbb{V}} + |\omega - \omega_0| < \varepsilon\}$;*
- (iii) *For all $s \in I$, we have $\int_{-\pi}^{\pi} \varphi_s^4 dx = \int_{-\pi}^{\pi} \varphi_0^4 dx$.*

**Proof.** The proof is similar to the one given in [2, Proposition 5] therefore we only give the main steps. Indeed, let $s_0 \in \left(\frac{1}{2}, 1\right)$ be fixed and consider $(\Phi_0, \omega_0) := (\varphi_0 + i0, \omega_0) \in \mathbb{V} \times \left(\frac{1}{2}, +\infty\right)$, where $\varphi_0 \in \mathbb{V}$ satisfies (1.6).

We define the mapping $\mathcal{G} : \mathbb{V} \times \left(\frac{1}{2}, +\infty\right) \times I_\delta \longrightarrow \mathbb{V} \times \mathbb{R}$ by

$$\mathcal{G}(\Phi, \omega, s) = \begin{bmatrix} \Phi - ((-\Delta)^s + \omega)^{-1}|\Phi|^2\Phi \\ \|\Phi\|^4_{L^4_{per}} - \|\Phi_0\|^4_{L^4_{per}} \end{bmatrix}, \tag{4.5}$$

where $I_\delta := [s_0, s_0 + \delta)$ with $\delta > 0$ will be chosen later. We note that $\mathcal{G}$ is a well-defined $C^1$-mapping ([24, Lemma E.1]) and $\mathcal{G}(\Phi_0, \omega_0, s_0) = (0, 0)$.

In particular, we see that the Fréchet derivative of $\mathcal{G}$ with respect $(\Phi, \omega)$ at $(\Phi_0, \omega_0, s_0)$ is given by

$$D_{\Phi,\omega}\mathcal{G}(\Phi_0, \omega_0, s_0) = \begin{bmatrix} 1 - ((-\Delta)^{s_0} + \omega_0)^{-1}\, 3\Phi_0^2 & ((-\Delta)^{s_0} + \omega_0)^{-2}\Phi_0^2 \\ 4(\Phi_0^3, \cdot)_{L^2_{per}} & 0 \end{bmatrix}.$$

Since $\varphi_0'$ is odd and $\mathrm{Ker}(\mathcal{L}_1) = [\varphi_0']$, we can show that $D_{\Phi,\omega}\mathcal{G}(\Phi_0, \omega_0, s_0)$ is invertible. By implicit function theorem, we guarantee the existence of a $C^1$-map

$$s \in I_\delta \longmapsto (\Phi_s, \omega_s) \in \mathbb{V} \times \left(\tfrac{1}{2}, +\infty\right), \tag{4.6}$$

defined over $I_\delta$, where $\delta > 0$ is small enough. Here $\Phi_s$ is defined in a neighbourhood of the point $\Phi_0 = \varphi_0 + 0i \in \mathbb{V}$ and this fact enables us to define, without loss of generality, that $\Phi_s := \varphi_s + 0i \in \mathbb{V}$. Thus, we can consider a local branch of solutions $(\varphi_s + 0i, \omega_s) \in \mathbb{V} \times \left(\frac{1}{2}, +\infty\right)$ for the equation (1.6) and parametrized by $s \in I_\delta$. $\square$

The next step is to consider the corresponding maximal extension of the branch $(\varphi_s, \omega_s) := (\varphi_s + 0i, \omega_s)$ given by $s \in [s_0, s_*)$, where

$$s_* := \sup\{q;\ s_0 < q < 1,\ (\varphi_s, \omega_s) \in C^1([s_0, q); \mathbb{V} \times \left(\tfrac{1}{2}, +\infty\right)) \text{ given by Proposition 4.9} \\ \text{and } (\varphi_s, \omega_s) \text{ satisfies (1.6) for } s \in [s_0, q)\}.$$

It is clear that $s_* \leqslant 1$ and it makes necessary to prove $s_* = 1$. To do so, we establish the following result:

**Proposition 4.10.** *Let $\{s_n\}_{n=1}^{n=+\infty} \subset \left(\frac{1}{2}, s_*\right)$ be a sequence such that $s_n \to s_*$. Furthermore, we assume that $\varphi_{s_n} \in \mathbb{V}$ are the corresponding solutions obtained in Proposition 4.9 with frequency of the wave given by $\omega_{s_n}$. Up to a subsequence, it follows that*

$$\varphi_{s_n} \to \varphi_* \text{ in } L^p_{per}(\mathbb{T}) \text{ and } \omega_{s_n} \to \omega_*,$$

*for all $p \geqslant 1$. Here, $\varphi_*$ satisfies equation the (1.6) where $\omega_* \in \left(\frac{1}{2}, +\infty\right)$ is the corresponding frequency of the wave. Moreover, the corresponding maximal branch $(\varphi_s, \omega_s) \in C^1\left([s_0, s_*); \mathbb{V} \times \left(\frac{1}{2}, +\infty\right)\right)$ extends to $s_* = 1$.*

**Proof.** The proof of this result is similar to [2, Proposition 6] and we omit the details. $\square$

**Proposition 4.11** *(Uniqueness of real minimizers). Let $s \in \left(\frac{1}{2}, 1\right)$ be fixed. If* $\mathrm{Ker}(\mathcal{L}_1) = [\varphi']$ *for all $\omega \in \left(\frac{1}{2}, +\infty\right)$, the real and even periodic minimizer obtained in Proposition 3.2 is unique.*

**Proof.** It follows by similar arguments as in [2, Proposition 7]. $\square$

## 5. Orbital stability

In this section, we present the orbital stability results. It is well known that (1.1) has two basic symmetries, namely, translation and rotation. If $u = u(x,t)$ is a solution of (1.1), so are $e^{-i\zeta}u$ and $u(x-r,t)$ for any $\zeta, r \in \mathbb{R}$. Considering $u = P + iQ \equiv (P, Q)$, we obtain that (1.1) is invariant under the transformations

$$S_1(\zeta)u := \begin{pmatrix} \cos\zeta & \sin\zeta \\ -\sin\zeta & \cos\zeta \end{pmatrix}\begin{pmatrix} P \\ Q \end{pmatrix} \tag{5.1}$$

and

$$S_2(r)u := \begin{pmatrix} P(\cdot - r, \cdot) \\ Q(\cdot - r, \cdot) \end{pmatrix}. \tag{5.2}$$

The actions $S_1$ and $S_2$ define unitary groups in $H^s_{per}$ with infinitesimal generators given by $S_1'(0)u := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} P \\ Q \end{pmatrix} = J\begin{pmatrix} P \\ Q \end{pmatrix}$ and $S_2'(0)u := \partial_x \begin{pmatrix} P \\ Q \end{pmatrix}$.

A standing wave solution as in (1.5) is given by $u(x,t) = e^{i\omega t}\varphi(x) = \begin{pmatrix} \varphi(x)\cos(\omega t) \\ \varphi(x)\sin(\omega t) \end{pmatrix}$. Since the equation (1.1) is invariant under the actions of $S_1$ and $S_2$, we define the orbit generated by $\Phi = (\varphi, 0)$ as

$$\mathcal{O}_\Phi = \left\{S_1(\zeta)S_2(r)\Phi;\, \zeta, r \in \mathbb{R}\right\} = \left\{\begin{pmatrix} \cos\zeta & \sin\zeta \\ -\sin\zeta & \cos\zeta \end{pmatrix}\begin{pmatrix} \varphi(\cdot - r) \\ 0 \end{pmatrix};\ \zeta, r \in \mathbb{R}\right\}.$$

The pseudometric $d$ in $H^s_{per}$ is given by $d(f,g) := \inf\{\|f - S_1(\zeta)S_2(r)g\|_{H^s_{per}};\ \ \zeta, r \in \mathbb{R}\}$. The distance between $f$ and $g$ is the distance between $f$ and the orbit generated by $g$ under the action of rotation and translation, so that $d(f, \Phi) = d(f, \mathcal{O}_\Phi)$.

We now present our notion of orbital stability.

**Definition 5.1.** Let $\Theta(x,t) = (\varphi(x)\cos(\omega t), \varphi(x)\sin(\omega t))$ be a standing wave for (1.1). We say that $\Theta$ is orbitally stable in $H^s_{per}$ provided that, given $\varepsilon > 0$, there exists $\delta > 0$ with the following property: if $u_0 \in H^s_{per}$ satisfies $\|u_0 - \Phi\|_{H^s_{per}} < \delta$, then the local solution $u(t)$ defined in the semi-interval $[0, +\infty)$ satisfies $d(u(t), \mathcal{O}_\Phi) < \varepsilon$, for all $t \geqslant 0$. Otherwise, we say that $\Theta$ is orbitally unstable in $H^s_{per}$.

**Proof of Theorem 1.1.** By Proposition 4.6, we see that $n(\mathcal{L}) = 1$ and $\mathrm{Ker}(\mathcal{L}) = [(\varphi', 0), (0, \varphi)]$ and these two basic facts are crucial to determine results of orbital stability/instability for periodic waves. Since both spectral properties are valid, the proof of orbital stability follows similarly as in [47, Theorem 4.17] but we need to take into account the result of global well-posedness as in Proposition 2.5 to prove the stability in terms of the two symmetries defined for the orbit $\mathcal{O}_\Phi$. For the orbital stability, we need to consider the Vakhitov–Kolokolov condition $\mathsf{q} > 0$ which is equivalent to consider $(\mathcal{L}_1\Psi, \Psi)_{L^2_{per}} < 0$, where $\Psi = -\frac{d}{d\omega}\varphi$, $\mathcal{L}_1\Psi = \varphi$ and $(\mathcal{L}_1\Psi, \varphi')_{L^2_{per}} = 0$. For the orbital instability in $H^s_{per,e}$, we first use the approach in [28] and the condition $\mathsf{q} < 0$ by considering the orbit $\mathcal{O}_\Phi$ having only one basic symmetry (namely, the orbit generated by the rotations only). As far as we know, the theory in [28] only requires that $\mathrm{n}(\mathcal{L}) = 1$ and $\mathrm{z}(\mathcal{L}) = 1$, so that we need to remove out one of the symmetries in Definition 5.1. Since the space $H^s_{per,e}$ is not invariant under translation and $\varphi'$ is odd, the pair $(\varphi', 0)$ can not be considered as an element of the subspace $\mathrm{Ker}(\mathcal{L})$ and thus, under this restriction, we have $\mathrm{Ker}(\mathcal{L}|_{L^2_{per,e}}) = [(0,\varphi)]$. Here $\mathcal{L}|_{L^2_{per,e}}$ denotes the restriction of $\mathcal{L}$ over the subspace of even functions $L^2_{per,e}$. It is clear that if the standing wave is orbitally unstable in a subspace $H^s_{per,e}$ of $H^s_{per}$, then it will also be unstable in the whole energy space $H^s_{per}$. The numerical approach determined below will be useful to decide the values of $s \in (\frac{1}{4}, 1]$ for which $\mathsf{q} > 0$ or $\mathsf{q} < 0$ in order to prove the orbital stability/instability. □

### 5.1. Numerical experiments - proof of Theorem 1.2

In this section we generate the periodic standing wave solutions of the fNLS equation by using the Petviashvili's iteration method. The method is widely used for the generation of travelling wave solutions ([21,22,41,49–51]). We refer to [38] for the numerical study on the fNLS equation where an iterative Newton method is used to construct the travelling wave solutions. Besides providing a numerical method in order to present the periodic single-lobe profile $\varphi$, our intention is to determine the sign of the quantity:

$$\mathsf{q} = \frac{d}{d\omega}\int_{-\pi}^{\pi} \varphi^2 dx. \tag{5.3}$$

Applying the Fourier transform to the equation (1.6) gives

$$\left(|\xi|^{2s} + \omega\right)\widehat{\varphi}(\xi) - \widehat{\varphi^3}(\xi) = 0, \quad \xi \in \mathbb{Z}. \tag{5.4}$$

An iterative algorithm for numerical calculation of $\widehat{\psi}(\xi)$ for the equation (5.4) can be proposed in the form

$$\widehat{\varphi}_{n+1}(\xi) = \frac{\widehat{\varphi^3_n}(\xi)}{|\xi|^{2s} + \omega}, \qquad n \in \mathbb{N} \tag{5.5}$$

where $\widehat{\varphi_n}(\xi)$ is the Fourier transform of $\varphi_n$ which is the $n^{\text{th}}$ iteration of the numerical solution. Here the solutions are constructed under the assumption

$$|\xi|^{2s} + \omega \neq 0. \tag{5.6}$$

Since the above algorithm is usually divergent, we finally present the Petviashvili's method as

$$\widehat{\varphi}_{n+1}(\xi) = \frac{(M_n)^\nu}{|\xi|^{2s} + \omega}\widehat{\varphi^3_n}(\xi) \tag{5.7}$$

by introducing the stabilizing factor

$$M_n = \frac{\left(((-\Delta)^s + \omega)\varphi_n, \varphi_n\right)_{L^2_{per}}}{(\varphi_n^3, \varphi_n)_{L^2_{per}}}, \qquad \varphi_n \in H^{2s}_{per}(\mathbb{T}). \tag{5.8}$$

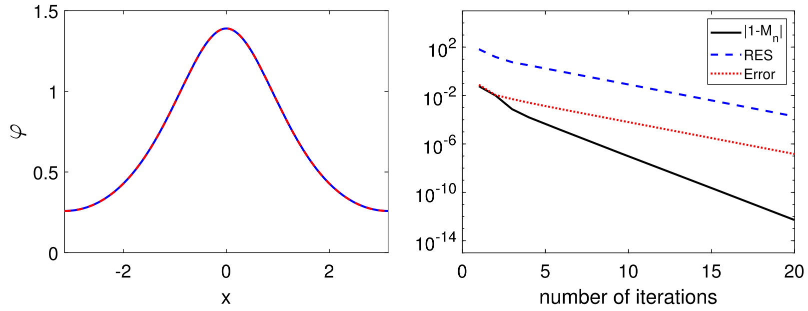


Fig. 5.1. The exact and the numerical solutions of the fNLS equation with the wave frequency $\omega = 1$ and the variation of $\mathrm{Error}(n)$, $|1 - M_n|$ and $RES$ with the number of iterations in semi-log scale.

Here, the free parameter $\nu$ is chosen as $3/2$ for the fastest convergence. The iterative process is controlled by the error between two consecutive iterations given by

$$\mathrm{Error}(n) = \|\varphi_n - \varphi_{n-1}\|_{L^\infty_{per}}$$

and the stabilization factor error given by $|1 - M_n|$. The residual error is determined by $RES(n) = \|\mathcal{S}\varphi_n\|_{L^\infty_{per}}$, where $\mathcal{S}\varphi = (-\Delta)^s\varphi + \omega\varphi - \varphi^3$.

The periodic standing wave solution of the fNLS equation with $s = 1$ is given in [4] as

$$\varphi(x) = \eta_1 \mathrm{dn}\left(\frac{\eta_1}{\sqrt{2}}x; \kappa\right), \tag{5.9}$$

where $\kappa^2 = \frac{\eta_1^2 - \eta_2^2}{\eta_1^2}$, $\quad \eta_1^2 - \eta_2^2 = 2\omega$, $\quad 0 < \eta_2 < \eta_1$. Here the fundamental period is $T_\varphi = \dfrac{2\sqrt{2}}{\eta_1}\mathrm{K}(\kappa)$ where $\mathrm{K}(\kappa)$ is the complete elliptic integral of first kind.

In order to test the accuracy of our scheme, we compare the exact solution (5.9) with the numerical solution obtained by using (3.26) as the initial guess. The space interval is $[-\pi, \pi]$ and number of grid points is chosen as $N = 2^{10}$. In the left panel of Fig. 5.1, we present the exact and numerical solutions for the frequency $\omega = 1$. As it is seen from the figure, the exact and the numerical solutions coincide. In the right panel of Fig. 5.1, the variations of three different errors with the number of iteration are presented. These results show that our numerical scheme captures the solution remarkably well.

The exact solutions of the fNLS equation are not known for $s \in (0, 1)$. In Fig. 5.2 we illustrate the periodic wave profiles for several values of $s \in (0, 1)$ with $\omega = 1$. The nonlinear term becomes dominant with decreasing values of $s \in (0, 1)$. Therefore, the wave steepens as expected.

In the rest of the numerical experiments, the sign of $\mathsf{q}$ in (5.3) is determined by investigating the value of $\|\varphi\|^2_{L^2_{per}}$ is increasing or decreasing with respect to $\omega$. The interval $\omega \in (1/2, 50]$ is discretized into 1000 subintervals. For each value of $\omega$, we generate the periodic wave profile by using the Petviashvili's iteration method on the interval $[-\pi, \pi]$ with $N = 2^{14}$. Then, we evaluate the value of $\|\varphi\|^2_{L^2_{per}}$.

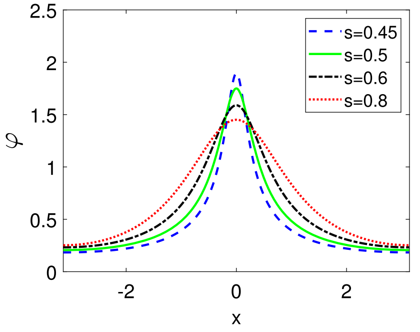


Fig. 5.2. Numerical wave profiles for various values of $s \in (0, 1)$ where $\omega = 1$.

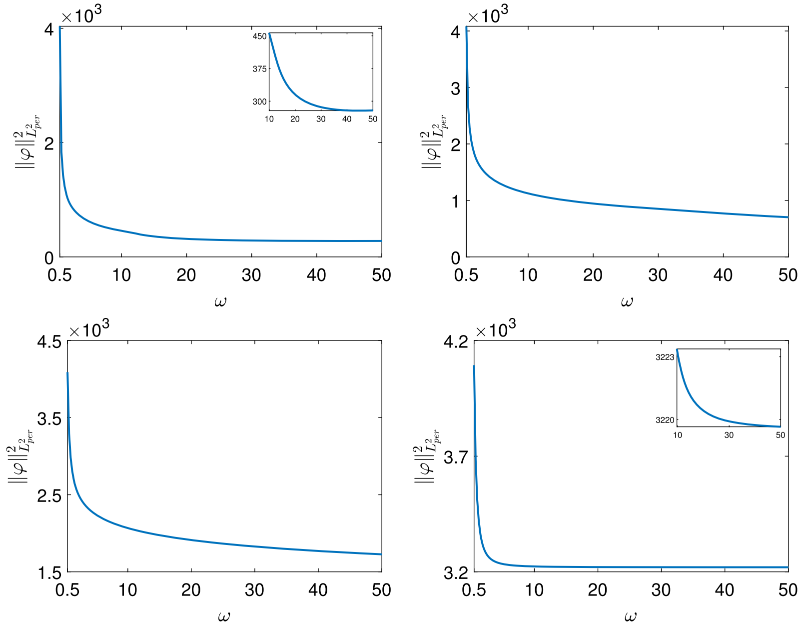


Fig. 5.3. The variation of $\|\varphi\|^2_{L^2_{per}}$ with $\omega$ for $s = 0.35$ (top left), $s = 0.4$ (top right), $s = 0.45$ (bottom left), $s = 0.5$ (bottom right).

In Fig. 5.3 we illustrate the variation of $\|\varphi\|^2_{L^2_{per}}$ with $\omega > \frac{1}{2}$ for $s = 0.35$, $s = 0.4$, $s = 0.45$ and $s = 0.5$. As it is seen from the figure, $\|\varphi\|^2_{L^2_{per}}$ is decreasing so that $\mathsf{q}$ is negative. Numerical results indicate that the periodic wave is orbitally unstable for $s \in (\frac{1}{4}, \frac{1}{2}]$.

The variation of $\|\varphi\|^2_{L^2_{per}}$ with $\omega > \frac{1}{2}$ for $s = 0.6$ and $s = 0.8$ is depicted in Fig. 5.4. Since $\|\varphi\|^2_{L^2_{per}}$ is increasing, $\mathsf{q}$ is positive. Therefore, the numerical results show the orbital stability of the periodic wave for $s \in [0.6, 1)$.

We have performed numerical experiments for several values of $s \in (0.5, 0.6)$. The numerical results indicate that there is a critical wave frequency $\omega_c$ such that $\mathsf{q}$ is negative for $\omega < \omega_c$

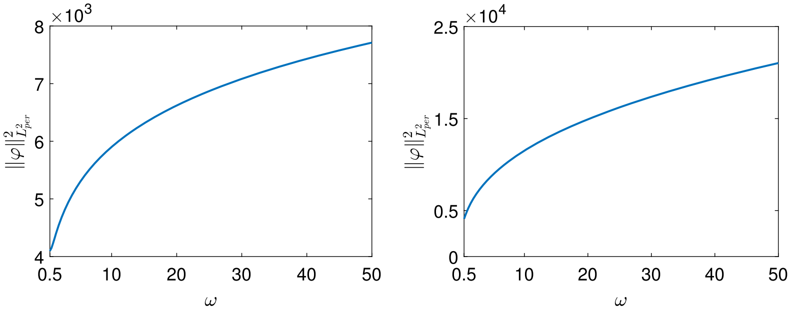

Fig. 5.4. The variation of $\|\varphi\|^2_{L^2_{per}}$ with $\omega$ for $s = 0.6$ (left panel) and $s = 0.8$ (right panel).

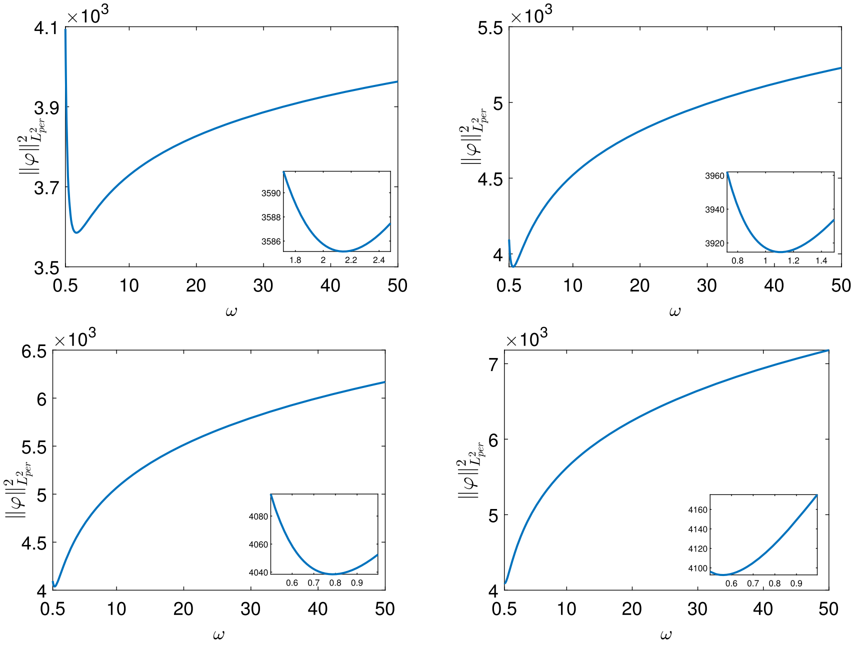

Fig. 5.5. The variation of $\|\varphi\|^2_{L^2_{per}}$ with $\omega$ for $s = 0.52$ (top left), $s = 0.55$ (top right), $s = 0.57$ (bottom left), $s = 0.59$ (bottom right).

and positive for $\omega > \omega_c$ for the values $s \in (0.5, 0.6)$. In Fig. 5.5, the variation of $\|\varphi\|^2_{L^2_{per}}$ with $\omega > \frac{1}{2}$ for $s = 0.52, 0.55, 0.57$ and $s = 0.59$ is presented. As it is seen from the figure $\|\varphi\|^2_{L^2_{per}}$ is decreasing up to the critical wave frequency $\omega_c$ and then it is increasing.

## Data availability

No data was used for the research described in the article.

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